Notes on the function gsw_rho_first_derivatives_CT_exact(SA,CT,p)

This function, **gsw_rho_first_derivatives_CT_exact**(SA,CT,p), evaluates the first derivatives of *in situ* density ρ with respect to Absolute Salinity, Conservative Temperature and pressure, with the input temperature being Conservative Temperature Θ. This function uses the full TEOS-10 Gibbs function $g(S_\mathrm{A},t,p)$ of IOC *et al*. (2010), being the sum of the IAPWS-09 and IAPWS-08 Gibbs functions.

This function evaluates the partial derivative of *in situ* density with respect to Absolute Salinity, $\partial\rho/\partial S_\mathrm{A}\big|_{\Theta,p}$ from the expression for β^Θ derived in appendix A.15 of the TEOS-10 Manual (IOC *et al*., 2010), namely Eqn. (A.15.10), that is,

$$\beta^\Theta = \frac{1}{\rho}\frac{\partial\rho}{\partial S_\mathrm{A}}\bigg|_{\Theta,p} = -\frac{g_{S_\mathrm{A}P}(S_\mathrm{A},t,p)}{g_P(S_\mathrm{A},t,p)} + \frac{g_{TP}(S_\mathrm{A},t,p)\left[g_{S_\mathrm{A}T}(S_\mathrm{A},t,p)-g_{S_\mathrm{A}}(S_\mathrm{A},\theta,0)/(T_0+\theta)\right]}{g_P(S_\mathrm{A},t,p)\,g_{TT}(S_\mathrm{A},t,p)}\,.$$

In this way the expression that is used to evaluate $\partial\rho/\partial S_\mathrm{A}\big|_{\Theta,p}$ is

$$\frac{\partial\rho}{\partial S_\mathrm{A}}\bigg|_{\Theta,p} = -\frac{g_{S_\mathrm{A}P}(S_\mathrm{A},t,p)}{\left(g_P(S_\mathrm{A},t,p)\right)^2} + \frac{g_{TP}(S_\mathrm{A},t,p)\left[g_{S_\mathrm{A}T}(S_\mathrm{A},t,p)-g_{S_\mathrm{A}}(S_\mathrm{A},\theta,0)/(T_0+\theta)\right]}{\left(g_P(S_\mathrm{A},t,p)\right)^2 g_{TT}(S_\mathrm{A},t,p)}\,.$$

The first steps in the code are to evaluate both the *in situ* temperature t and the potential temperature θ with respect to $p_\mathrm{r} = 0\,\mathrm{dbar}$. Both the terms $g_{S_\mathrm{A}T}(S_\mathrm{A},t,p)$ and $g_{S_\mathrm{A}}(S_\mathrm{A},\theta,0)/(T_0+\theta)$ in the second part of this equation contain logarithmic singularities in the square root of Absolute Salinity, but these singularities exactly cancel in the square bracket in Eqn. (A.15.10). Hence, in the present code this square bracket is not simply calculated by calling the appropriate derivatives of the Gibbs function, but rather, the polynomials representing both $g_{S_\mathrm{A}T}(S_\mathrm{A},t,p)$ and $g_{S_\mathrm{A}}(S_\mathrm{A},\theta,0)$ are incorporated into the present function **gsw_rho_first_derivatives_CT_exact**(SA,t,p), and the logarithm terms are deliberately excluded. In this way, this function can be used with the input value of S_A being exactly zero.

This function evaluates the partial derivative of *in situ* density with respect to Conservative Temperature, $\partial\rho/\partial\Theta\big|_{S_\mathrm{A},p}$, using the expression for α^Θ derived in appendix A.15 of the TEOS-10 Manual (IOC *et al*., 2010), namely Eqn. (A.15.4), that is,

$$\alpha^\Theta = -\frac{1}{\rho}\frac{\partial\rho}{\partial\Theta}\bigg|_{S_\mathrm{A},p} = \frac{1}{v}\frac{\partial v}{\partial\Theta}\bigg|_{S_\mathrm{A},p} = \frac{1}{v}\frac{\partial v}{\partial T}\bigg|_{S_\mathrm{A},p}\left(\frac{\partial\Theta}{\partial T}\bigg|_{S_\mathrm{A},p}\right)^{-1}$$
$$= -\frac{g_{TP}(S_\mathrm{A},t,p)}{g_P(S_\mathrm{A},t,p)}\,\frac{c_p^0}{(T_0+\theta)\,g_{TT}(S_\mathrm{A},t,p)}\,.$$

That is, the present code evaluates $\partial\rho/\partial\Theta\big|_{S_\mathrm{A},p}$ from

$$\frac{\partial\rho}{\partial\Theta}\bigg|_{S_\mathrm{A},p} = \frac{g_{TP}(S_\mathrm{A},t,p)}{\left(g_P(S_\mathrm{A},t,p)\right)^2}\,\frac{c_p^0}{(T_0+\theta)\,g_{TT}(S_\mathrm{A},t,p)}\,.$$

This function evaluates $\partial\rho/\partial P\big|_{S_\mathrm{A},\Theta}$ using Eqn. (2.16.1) of the TEOS-10 Manual (IOC *et al*., 2010), for the isentropic (and isohaline) compressibility κ, repeated here,

$$\kappa = \rho^{-1} \frac{\partial \rho}{\partial P}\bigg|_{S_A,\eta} = -v^{-1} \frac{\partial v}{\partial P}\bigg|_{S_A,\eta} = \rho^{-1} \frac{\partial \rho}{\partial P}\bigg|_{S_A,\theta} = \rho^{-1} \frac{\partial \rho}{\partial P}\bigg|_{S_A,\Theta} \\ = \frac{\left(g_{TP}^2 - g_{TT} g_{PP}\right)}{g_P \, g_{TT}} . \qquad (2.16.1)$$

That is, the present code evaluates $\partial\rho/\partial P\big|_{S_A,\Theta}$ from

$$\frac{\partial \rho}{\partial P}\bigg|_{S_A,\Theta} = \frac{\left(g_{TP}^2 - g_{TT} g_{PP}\right)}{\left(g_P\right)^2 g_{TT}} ,$$

where all the partial derivatives of the Gibbs function are evaluated at $\left(S_A, t, p\right)$. The pressure derivative is taken with respect to pressure in Pa rather than with resepct to pressure in dbar. This is done so that this pressure derivative of ρ is compatible with straightforward evaluation of the isentropic compressibility ($\kappa = \rho^{-1}\partial\rho/\partial P\big|_{S_A,\Theta}$) and the sound speed c (since $\partial\rho/\partial P\big|_{S_A,\Theta} = c^{-2}$).

References

IAPWS, 2008: Release on the IAPWS Formulation 2008 for the Thermodynamic Properties of Seawater. The International Association for the Properties of Water and Steam. Berlin, Germany, September 2008, available from www.iapws.org. This Release is referred to in the text as **IAPWS-08**.

IAPWS, 2009: Supplementary Release on a Computationally Efficient Thermodynamic Formulation for Liquid Water for Oceanographic Use. The International Association for the Properties of Water and Steam. Doorwerth, The Netherlands, September 2009, available from http://www.iapws.org. This Release is referred to in the text as **IAPWS-09**.

IOC, SCOR and IAPSO, 2010: *The international thermodynamic equation of seawater – 2010: Calculation and use of thermodynamic properties*. Intergovernmental Oceanographic Commission, Manuals and Guides No. 56, UNESCO (English), 196 pp. Available from http://www.TEOS-10.org

Here follows sections 2.16, 2.17 and appendix A.15 of the TEOS-10 Manual (IOC *et al.*, 2010).

2.16 Isentropic and isohaline compressibility

When the entropy and Absolute Salinity are held constant while the pressure is changed, the isentropic and isohaline compressibility κ is obtained:

$$\kappa = \kappa\left(S_A, t, p\right) = \rho^{-1} \frac{\partial \rho}{\partial P}\bigg|_{S_A,\eta} = -v^{-1} \frac{\partial v}{\partial P}\bigg|_{S_A,\eta} = \rho^{-1} \frac{\partial \rho}{\partial P}\bigg|_{S_A,\theta} = \rho^{-1} \frac{\partial \rho}{\partial P}\bigg|_{S_A,\Theta} \\ = \frac{\left(g_{TP}^2 - g_{TT} g_{PP}\right)}{g_P \, g_{TT}} . \qquad (2.16.1)$$

The isentropic and isohaline compressibility κ is sometimes called simply the isentropic compressibility (or sometimes the "adiabatic compressibility"), on the unstated understanding that there is also no transfer of salt during the isentropic or adiabatic change in pressure. The isentropic and isohaline compressibility of seawater κ produced by both the SIA and GSW software libraries (appendices M and N) has units of Pa^{-1}.

2.17 Sound speed

The speed of sound in seawater c is given by

$$c = c(S_\mathrm{A},t,p) = \left(\partial P/\partial\rho|_{S_\mathrm{A},\eta}\right)^{0.5} = (\rho\kappa)^{-0.5} = g_P\left(g_{TT}\big/\left[g_{TP}^2 - g_{TT}\,g_{PP}\right]\right)^{0.5}. \qquad (2.17.1)$$

Note that in these expressions in Eqn. (2.17.1), since sound speed is in $\mathrm{m\,s^{-1}}$ and density has units of $\mathrm{kg\,m^{-3}}$ it follows that the pressure of the partial derivatives must be in Pa and the isentropic compressibility κ must have units of Pa^{-1}. The sound speed c produced by both the SIA and the GSW software libraries (appendices M and N) has units of $\mathrm{m\,s^{-1}}$.

A.15 Derivation of the expressions for α^θ, β^θ, α^Θ and β^Θ

This appendix derives the expressions in Eqns. (2.18.2) – (2.18.3) and (2.19.2) – (2.19.3) for the thermal expansion coefficients α^θ and α^Θ and the haline contraction coefficients β^θ and β^Θ.

In order to derive Eqn. (2.18.2) for α^θ we first need an expression for $\partial\theta/\partial T|_{S_\mathrm{A},p}$. This is found by differentiating with respect to *in situ* temperature the entropy equality $\eta(S_\mathrm{A},t,p) = \eta(S_\mathrm{A},\theta[S_\mathrm{A},t,p,p_\mathrm{r}],p_\mathrm{r})$ which defines potential temperature, obtaining

$$\left.\frac{\partial\theta}{\partial T}\right|_{S_\mathrm{A},p} = \frac{\eta_T(S_\mathrm{A},t,p)}{\eta_T(S_\mathrm{A},\theta,p_\mathrm{r})} = \frac{g_{TT}(S_\mathrm{A},t,p)}{g_{TT}(S_\mathrm{A},\theta,p_\mathrm{r})} = \frac{(T_0+\theta)}{(T_0+t)}\frac{c_p(S_\mathrm{A},t,p)}{c_p(S_\mathrm{A},\theta,p_\mathrm{r})}. \qquad (\mathrm{A}.15.1)$$

This is then used to obtain the desired expression Eqn. (2.18.2) for α^θ as follows

$$\alpha^\theta = \frac{1}{v}\left.\frac{\partial v}{\partial\theta}\right|_{S_\mathrm{A},p} = \frac{1}{v}\left.\frac{\partial v}{\partial T}\right|_{S_\mathrm{A},p}\left(\left.\frac{\partial\theta}{\partial T}\right|_{S_\mathrm{A},p}\right)^{-1} = \frac{g_{TP}(S_\mathrm{A},t,p)}{g_P(S_\mathrm{A},t,p)}\,\frac{g_{TT}(S_\mathrm{A},\theta,p_\mathrm{r})}{g_{TT}(S_\mathrm{A},t,p)}. \qquad (\mathrm{A}.15.2)$$

In order to derive Eqn. (2.18.3) for α^Θ we first need an expression for $\partial\Theta/\partial t|_{S_\mathrm{A},p}$. This is found by differentiating with respect to *in situ* temperature the entropy equality $\eta(S_\mathrm{A},t,p) = \hat\eta(S_\mathrm{A},\Theta[S_\mathrm{A},t,p])$ obtaining

$$\left.\frac{\partial\Theta}{\partial T}\right|_{S_\mathrm{A},p} = \eta_T(S_\mathrm{A},t,p)\left.\frac{\partial\Theta}{\partial\eta}\right|_{S_\mathrm{A}} = -(T_0+\theta)\frac{g_{TT}(S_\mathrm{A},t,p)}{c_p^0} = \frac{(T_0+\theta)}{(T_0+t)}\frac{c_p(S_\mathrm{A},t,p)}{c_p^0}, \qquad (\mathrm{A}.15.3)$$

where the second part of this equation has used Eqn. (A.12.4) for $\Theta_\eta|_{S_\mathrm{A}}$. This is then used to obtain the desired expression Eqn. (2.18.3) for α^Θ as follows

$$\alpha^\Theta = \frac{1}{v}\left.\frac{\partial v}{\partial\Theta}\right|_{S_\mathrm{A},p} = \frac{1}{v}\left.\frac{\partial v}{\partial T}\right|_{S_\mathrm{A},p}\left(\left.\frac{\partial\Theta}{\partial T}\right|_{S_\mathrm{A},p}\right)^{-1} = -\frac{g_{TP}(S_\mathrm{A},t,p)}{g_P(S_\mathrm{A},t,p)}\,\frac{c_p^0}{(T_0+\theta)\,g_{TT}(S_\mathrm{A},t,p)}. \qquad (\mathrm{A}.15.4)$$

In order to derive Eqn. (2.19.2) for β^θ we first need an expression for $\partial\theta/\partial S_\mathrm{A}|_{T,p}$. This is found by differentiating with respect to Absolute Salinity the entropy equality $\eta(S_\mathrm{A},t,p) = \eta(S_\mathrm{A},\theta[S_\mathrm{A},t,p,p_\mathrm{r}],p_\mathrm{r})$ which defines potential temperature, obtaining

$$\begin{aligned}
\left.\frac{\partial\theta}{\partial S_\mathrm{A}}\right|_{T,p} &= \theta_\eta|_{S_\mathrm{A}}\left[\eta_{S_\mathrm{A}}(S_\mathrm{A},t,p) - \eta_{S_\mathrm{A}}(S_\mathrm{A},\theta,p_\mathrm{r})\right] \\
&= \frac{(T_0+\theta)}{c_p(S_\mathrm{A},\theta,p_\mathrm{r})}\left[\mu_T(S_\mathrm{A},\theta,p_\mathrm{r}) - \mu_T(S_\mathrm{A},t,p)\right] \qquad (\mathrm{A}.15.5) \\
&= \left[g_{S_\mathrm{A}T}(S_\mathrm{A},t,p) - g_{S_\mathrm{A}T}(S_\mathrm{A},\theta,p_\mathrm{r})\right]\big/g_{TT}(S_\mathrm{A},\theta,p_\mathrm{r}),
\end{aligned}$$

where Eqns. (A.12.5) and (A.12.7) have been used with a general reference pressure p_{r} rather than with $p_{\mathrm{r}} = 0$. By differentiating $\rho = \hat{\rho}\left(S_{\mathrm{A}}, \theta\left[S_{\mathrm{A}}, t, p, p_{\mathrm{r}}\right], p\right)$ with respect to Absolute Salinity it can be shown that (Gill (1982), McDougall (1987a))

$$\beta^{\theta} = \frac{1}{\rho}\frac{\partial\rho}{\partial S_{\mathrm{A}}}\bigg|_{\theta,p} = \frac{1}{\rho}\frac{\partial\rho}{\partial S_{\mathrm{A}}}\bigg|_{T,p} + \alpha^{\theta}\frac{\partial\theta}{\partial S_{\mathrm{A}}}\bigg|_{T,p}, \tag{A.15.6}$$

and using Eqn. (A.15.5) we arrive at the desired expression Eqn. (2.19.2) for β^{θ}

$$\beta^{\theta} = -\frac{g_{S_{\mathrm{A}}P}\left(S_{\mathrm{A}},t,p\right)}{g_{P}\left(S_{\mathrm{A}},t,p\right)} + \frac{g_{TP}\left(S_{\mathrm{A}},t,p\right)\left[g_{S_{\mathrm{A}}T}\left(S_{\mathrm{A}},t,p\right) - g_{S_{\mathrm{A}}T}\left(S_{\mathrm{A}},\theta,p_{\mathrm{r}}\right)\right]}{g_{P}\left(S_{\mathrm{A}},t,p\right)g_{TT}\left(S_{\mathrm{A}},t,p\right)}\ . \tag{A.15.7}$$

Note that the terms in the natural logarithm of the square root of Absolute Salinity cancel from the two parts of the square brackets in Eqns. (A.15.5) and (A.15.7).

In order to derive Eqn. (2.19.3) for β^{Θ} we first need an expression for $\partial\Theta/\partial S_{\mathrm{A}}\big|_{T,p}$. This is found by differentiating with respect to Absolute Salinity the entropy equality $\eta\left(S_{\mathrm{A}},t,p\right) = \hat{\eta}\left(S_{\mathrm{A}},\Theta\left[S_{\mathrm{A}},t,p\right]\right)$ obtaining (using Eqns. (A.12.4) and (A.12.8))

$$\begin{aligned}\frac{\partial\Theta}{\partial S_{\mathrm{A}}}\bigg|_{T,p} &= \Theta_{\eta}\big|_{S_{\mathrm{A}}}\left[\eta_{S_{\mathrm{A}}}\left(S_{\mathrm{A}},t,p\right) - \hat{\eta}_{S_{\mathrm{A}}}\big|_{\Theta}\right] \\ &= \left[\mu\left(S_{\mathrm{A}},\theta,0\right) - \left(T_0+\theta\right)\mu_{T}\left(S_{\mathrm{A}},t,p\right)\right]\big/c_p^0 \\ &= \left[g_{S_{\mathrm{A}}}\left(S_{\mathrm{A}},\theta,0\right) - \left(T_0+\theta\right)g_{S_{\mathrm{A}}T}\left(S_{\mathrm{A}},t,p\right)\right]\big/c_p^0\ .\end{aligned} \tag{A.15.8}$$

Differentiating $\rho = \hat{\rho}\left(S_{\mathrm{A}},\Theta\left[S_{\mathrm{A}},t,p\right],p\right)$ with respect to Absolute Salinity leads to

$$\beta^{\Theta} = \frac{1}{\rho}\frac{\partial\rho}{\partial S_{\mathrm{A}}}\bigg|_{\Theta,p} = \frac{1}{\rho}\frac{\partial\rho}{\partial S_{\mathrm{A}}}\bigg|_{T,p} + \alpha^{\Theta}\frac{\partial\Theta}{\partial S_{\mathrm{A}}}\bigg|_{T,p}, \tag{A.15.9}$$

and using Eqn. (A.15.8) we arrive at the desired expression (2.19.3) for β^{Θ} namely

$$\beta^{\Theta} = -\frac{g_{S_{\mathrm{A}}P}\left(S_{\mathrm{A}},t,p\right)}{g_{P}\left(S_{\mathrm{A}},t,p\right)} + \frac{g_{TP}\left(S_{\mathrm{A}},t,p\right)\left[g_{S_{\mathrm{A}}T}\left(S_{\mathrm{A}},t,p\right) - g_{S_{\mathrm{A}}}\left(S_{\mathrm{A}},\theta,0\right)\big/\left(T_0+\theta\right)\right]}{g_{P}\left(S_{\mathrm{A}},t,p\right)g_{TT}\left(S_{\mathrm{A}},t,p\right)}\ . \tag{A.15.10}$$

Note that the terms in the natural logarithm of the square root of Absolute Salinity cancel from the two parts of the square brackets in Eqns. (A.15.8) and (A.15.10).