Notes on the function gsw_geo_strf_dyn_height(SA,CT,p,p_ref)

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This function, **gsw_geo_strf_dyn_height**(SA,CT,p,p_ref) evaluates the dynamic height anomaly $\Psi\left(S_\mathrm{A},\Theta,p,p_\mathrm{ref}\right)$ relative to a reference pressure p_ref. It uses the computationally-efficient 48-term rational function expression of McDougall *et al*. (2011) for the specific volume of seawater $\hat{v}\left(S_\mathrm{A},\Theta,p\right)$ in terms of Absolute Salinity S_A, Conservative Temperature Θ and pressure p. This 48-term rational function expression for density is discussed in McDougall and Barker (2013) and in appendix A.30 and appendix K of the TEOS-10 Manual (IOC *et al*. (2010)). For dynamical oceanography we may take the 48-term rational function expression for density as essentially reflecting the full accuracy of TEOS-10. The dynamic height anomaly Ψ is also often called the "geopotential anomaly".

The input variables SA,CT,p are either a single vertical cast or a series of such vertical casts. The input variable p_ref is a single positive scalar reference pressure (in dbar). When p_ref is zero, **gsw_geo_strf_dyn_height**(SA,CT,p,p_ref) returns the dynamic height anomaly as defined in Eqn. (3) below. We will discuss this case first, and then go on to discuss how the code incorporates a general reference pressure p_ref.

The dynamic height anomaly is defined as the pressure integral of the specific volume anomaly $\hat{\delta}\left(S_\mathrm{A},\Theta,p\right)$ which we choose to define with respect to $S_\mathrm{SO}\equiv 35.165\,04\ \mathrm{g\ kg^{-1}}$ and $\Theta=0°\mathrm{C}$ as

$$\hat{\delta}\left(S_\mathrm{A},\Theta,p\right) = \hat{v}\left(S_\mathrm{A},\Theta,p\right) - \hat{v}\left(S_\mathrm{SO},0°\mathrm{C},p\right). \tag{1}$$

The thermodynamic identity

$$h_P\big|_{S_\mathrm{A},\Theta} = \hat{h}_P = \hat{v}, \tag{2}$$

is used to calculate the dynamic height anomaly Ψ according to

$$\begin{aligned}\Psi\left(S_\mathrm{A}[p],\Theta[p],p\right) &= -\int_{P_0}^{P}\hat{\delta}\left(S_\mathrm{A}[p'],\Theta[p'],p'\right)dP' \\ &= -\int_{P_0}^{P}\hat{v}\left(S_\mathrm{A}[p'],\Theta[p'],p'\right)dP' + \int_{P_0}^{P}\hat{v}\left(S_\mathrm{SO},\Theta=0°\mathrm{C},p'\right)dP' \\ &= -\int_{P_0}^{P}\hat{v}\left(S_\mathrm{A}[p'],\Theta[p'],p'\right)dP' + \hat{h}\left(S_\mathrm{SO},\Theta=0°\mathrm{C},p\right).\end{aligned} \tag{3}$$

Note that the lower limit of the pressure integral of $\hat{v}\left(S_\mathrm{SO},0°\mathrm{C},p'\right)$ is $\hat{h}\left(S_\mathrm{SO},\Theta=0°\mathrm{C},0\,\mathrm{dbar}\right)$ which is zero (being c_p^0 times $\Theta=0°\mathrm{C}$). Note also that the pressure derivative in Eqn. (2) and the pressure integral in Eqn. (3) are both done with pressure increments measured in Pa, not dbar. This ensures that specific volume, enthalpy and dynamic height anomaly retain their usual units of $\mathrm{m^3\ kg^{-1}}$, $\mathrm{J\ kg^{-1}}$and $\mathrm{m^2\ s^{-2}}$ $(=\mathrm{J\ kg^{-1}})$ respectively. The code operates by evaluating the last line of Eqn. (3) and the enthalpy $\hat{h}\left(S_\mathrm{SO},\Theta=0°\mathrm{C},0\,\mathrm{dbar}\right)$ is found from the library function **gsw_enthalpy_SSO_0_p**(p).

This present function, **gsw_geo_strf_dyn_height**, evaluates the pressure integral of specific volume using S_A and Θ "interpolated" with respect to pressure using a time-tested scheme based on the method of Reiniger and Ross (1968). Our version of the Reiniger and Ross method uses a weighted mean of (i) values obtained from linear interpolation of the two nearest data points, and (ii) a linear extrapolation of the pairs of data above and below. The results of this method resemble the use of a cubic spline, and was developed for the construction of the CSIRO Atlas of Regional Seas (Ridgway *et al*., 2002). This method of "interpolation" between the input "bottles" of (SA,CT,p) is used to realistically construct finely-resolved vertical profiles of S_A and Θ with a vertical (pressure) spacing between adjacent "bottles" of no more than $1\,\mathrm{dbar}$. This finely-resolved vertical profile is then used to evaluate the vertical (pressure) integral of specific volume in the last line of Eqn. (3), so that the integration is done over vertical intervals no larger than $1x10^4$ Pa $(1\,\text{dbar})$. This Reiniger and Ross "interpolation" is considerably more sophisticated than linear interpolation, and is done for two reasons; first to obtain more realistic vertical profiles of S_A and Θ than given by a simple piece-wise linear profile, and second, to avoid inaccuracies caused by the nonlinear nature of specific volume as a function of S_A, Θ and p. Only a "rough & ready" cowboy/cowgirl oceanographer would resort to linear interpolation.

So far we have been considering the special case where $p_\mathrm{ref} = 0\,\text{dbar}$. In this case the dynamic height anomaly is the streamfunction for the difference between the geostrophic velocity at pressure p to that at p = 0 dbar. It is more common in physical oceanography to select a deep reference pressure so that the streamfunction represents the difference between the geostrophic velocity at p to that on a deep reference pressure p_ref which is commonly 1500 dbar or 2000 dbar. This function, **gsw_geo_strf_dyn_height,** achieves this by subtracting the value of dynamic height anomaly at p_ref from the value of dynamic height anomaly of Eqn. (3) at the general pressure p. This value of $\Psi\left(p_\mathrm{ref}\right)$ is accurately evaluated internally in this function using Eqn. (3) at *exactly* this p_ref value of pressure. The geostrophic streamfunction for the flow at pressure p relative to the flow at p_ref is then given in terms of the difference between two values of dynamic height anomaly (from Eqn. (3) above), namely

$$\begin{aligned}
&\Psi\left(S_\mathrm{A}[p],\,\Theta[p],\,p\right) \;-\; \Psi\left(S_\mathrm{A}[p_\mathrm{ref}],\,\Theta[p_\mathrm{ref}],\,p_\mathrm{ref}\right) \\
&\qquad = -\int_{P_\mathrm{ref}}^{P} \hat{\delta}\left(S_\mathrm{A}[p'],\Theta[p'],p'\right) dP' \\
&\qquad = -\int_{P_\mathrm{ref}}^{P} \hat{v}\left(S_\mathrm{A}[p'],\Theta[p'],p'\right) dP' \;+\; \int_{P_\mathrm{ref}}^{P} \hat{v}\left(S_\mathrm{SO},\Theta=0°\mathrm{C},p'\right) dP' \\
&\qquad = -\int_{P_\mathrm{ref}}^{P} \hat{v}\left(S_\mathrm{A}[p'],\Theta[p'],p'\right) dP' \;+\; \hat{h}\left(S_\mathrm{SO},\Theta=0°\mathrm{C},\,p\right) \;-\; \hat{h}\left(S_\mathrm{SO},\Theta=0°\mathrm{C},\,p_\mathrm{ref}\right),
\end{aligned} \tag{4}$$

and this is what is returned as the output of **gsw_geo_strf_dyn_height**(SA,CT,p,p_ref). The function actually evaluates this using the top line of Eqn. (4), (i.e. the left-hand side of Eqn. (4)).

References

IOC, SCOR and IAPSO, 2010: *The international thermodynamic equation of seawater – 2010: Calculation and use of thermodynamic properties*. Intergovernmental Oceanographic Commission, Manuals and Guides No. 56, UNESCO (English), 196 pp. Available from http://www.TEOS-10.org See section 3.27 and appendix A.30.

McDougall T. J. and P. M. Barker, 2013: A computationally efficient 48-term expression for the density of seawater in terms of Conservative Temperature, and related properties of seawater. To be submitted to *Journal of Atmospheric and Ocean Technology.*

Reiniger, R. F. and C. K. Ross, 1968: A method of interpolation with application to oceanographic data. *Deep-Sea Res*. **15**, 185-193.

Ridgway K. R., J. R. Dunn, and J. L. Wilkin, 2002: Ocean interpolation by four-dimensional least squares -Application to the waters around Australia, *J. Atmos. Ocean. Tech*., **19**, 1357-1375.

Here follows section 3.27 of the TEOS-10 Manual (IOC *et al*. (2010)).

3.27 Dynamic height anomaly

The dynamic height anomaly Ψ with respect to the sea surface is given by

$$\Psi = -\int_{P_0}^{P} \hat{\delta}\big(S_{\mathrm{A}}[p'],\Theta[p'],p'\big)\,dP', \quad \text{where} \quad \hat{\delta}(S_{\mathrm{A}},\Theta,p) = \hat{v}(S_{\mathrm{A}},\Theta,p) - \hat{v}(S_{SO},0°\mathrm{C},p). \qquad (3.27.1)$$

This is the geostrophic streamfunction for the flow at pressure P with respect to the flow at the sea surface and $\hat{\delta}$ is the specific volume anomaly. Thus the two-dimensional gradient of Ψ in the P pressure surface is simply related to the difference between the horizontal geostrophic velocity $\mathbf{v}$ at P and at the sea surface $\mathbf{v}_0$ according to

$$\mathbf{k}\times\nabla_P\Psi \;=\; f\mathbf{v} - f\mathbf{v}_0 . \qquad (3.27.2)$$

Dynamic height anomaly is also commonly called the "geopotential anomaly". The specific volume anomaly, $\hat{\delta}$ in the vertical integral in Eqn. (3.27.1) could be replaced with specific volume $\hat{v}$ without affecting the isobaric gradient of the resulting streamfunction. That is, this substitution would not affect Eqn. (3.27.2) because the additional term is a function only of pressure. Traditionally it was important to use specific volume anomaly in preference to specific volume as it was more accurate with computer code which worked with single-precision variables. Since computers now regularly employ double-precision, this issue has been overcome and consequently either $\hat{\delta}$ or $\hat{v}$ could be used in the integrand of Eqn. (3.27.1), so making it either the "dynamic height anomaly" or the "dynamic height". As in the case of Eqn. (3.24.2), so also the dynamic height anomaly Eqn. (3.27.1) has not assumed that the gravitational acceleration is constant and so Eqn. (3.27.2) applies even when the gravitational acceleration is taken to vary in the vertical.

The dynamic height anomaly Ψ should be quoted in units of $\mathrm{m^2\,s^{-2}}$. These are the units in which the GSW Toolbox (appendix N) outputs dynamic height anomaly in the function **gsw_geo_strf_dyn_height**(SA,CT,p,p_ref). When the last argument of this function, p_ref, is other than zero, the function returns the dynamic height anomaly with respect to a (deep) reference pressure p_ref, rather than with respect to P_0 (i.e. zero dbar sea pressure) as in Eqn. (3.27.1). In this case the lateral gradient of the streamfunction represents the geostrophic velocity difference relative to the (deep) p_{ref} pressure surface, that is,

$$\mathbf{k}\times\nabla_P\Psi \;=\; f\mathbf{v} - f\mathbf{v}_{\mathrm{ref}} . \qquad (3.27.3)$$

Note that the integration in Eqn. (3.27.1) of specific volume anomaly with pressure must be done with pressure in Pa (not dbar) in order to have the resultant isobaric gradient, $\nabla_P\Psi$, in the usual units, being the product of the Coriolis parameter (units of $\mathrm{s^{-1}}$) and the velocity (units of $\mathrm{m\,s^{-1}}$).