Notes on the GSW code

gsw_z_from_p

for calculating height *z* from pressure *p*

Height z is measured positive upwards, so it is negative in the ocean. First, note that we use the following version of specific volume anomaly,

$$\delta = \hat{v}\big(S_A,\Theta,p\big) - \hat{v}\big(S_{SO},0°\text{C},p\big). \tag{1}$$

That is, the reference Absolute Salinity is the Absolute Salinity of the Standard Ocean, $S_{SO} \equiv 35.165\,04 \text{ g kg}^{-1}$, and the reference "temperature" is a fixed value of Conservative Temperature of zero degrees Celsius. Dynamic height anomaly Ψ is then defined by Eqn. (3.27.1) of IOC *et al*. (2010) as follows

$$\Psi \;=\; -\int_{P_0}^{P} \delta\big(p'\big)\, dP', \tag{2}$$

where $P_0 = 101\,325\,\text{Pa}$ is the standard atmosphere pressure.

The vertical integral of the hydrostatic equation ($P_z = -g\,\rho$ or $g = -v\,P_z$) is (from Eqn. (3.32.3) of the TEOS-10 Manual (IOC *et al*. (2010)))

$$\begin{aligned}\int_0^z g\big(z'\big)\,dz' \;&=\; \Phi^0 - \int_{P_0}^{P} v\big(p'\big)\,dP' \;=\; -\int_{P_0}^{P} \hat{v}\big(S_{SO},0°\text{C},p'\big)\,dP' + \Psi + \Phi^0 \\ &= -\,\hat{h}\big(S_{SO},0°\text{C},p\big) + \Psi + \Phi^0,\end{aligned} \tag{3.23.3}$$

Here Φ^0 is the geopotential at zero sea pressure on this vertical cast. We use the 48-term based expression for enthalpy (Eqn. (A.30.6) of the TEOS-10 Manual), recognizing that because $\Theta = 0°\text{C}$ many of the coefficients on pages 132 of the TEOS-10 Manual are zero, so the evaluation of Eqn. (A.30.6) is less computationally expensive than it may appear. The library function **gsw_enthalpy_SSO_0_p**(p) is used to evaluate $\hat{h}^{48}\big(S_{SO},\,0°\text{C},p\big)$ efficiently at these fixed values of Absolute Salinity and Conservative Temperature.

Writing the gravitational acceleration of Eqn. (D.3) of IOC *et al*. (2010) as

$$g \;=\; g\big(\phi,z\big) \;=\; g\big(\phi,0\big)\big(1-\gamma z\big), \tag{4}$$

we see that Eqn. (3.32.3) becomes

$$\hat{h}^{48}\big(S_{SO},\,0°\text{C},p\big) - \Psi - \Phi^0 + g\big(\phi,0\big)\Big(z - \tfrac{1}{2}\gamma\, z^2\Big) \;=\; 0\;. \tag{5}$$

When the **gsw_z_from_p** code is called with two arguments, as in **gsw_z_from_p**(p,lat), $\Psi + \Phi^0$ is ignored in Eqn. (5) and this quadratic expression is solved for the height z. We do this using the standard quadratic solution equation, but for z^{-1}. This is done so that the result is accurate as pressure tends to zero, and so that the answer also converges to the correct solution when the quadratic term γ tends to zero (since there may be some applications where it is preferable to assume that the gravitational acceleration is depth-independent). Hence we evaluate z from the equation

$$z \;=\; -\,\frac{2\Big(\hat{h}^{48}\big(S_{SO},\,0°\text{C},p\big) - \Psi - \Phi^0\Big)}{g\big(\phi,0\big) + \sqrt{g^2\big(\phi,0\big) \,+\, 2\gamma\, g\big(\phi,0\big)\Big(\hat{h}^{48}\big(S_{SO},\,0°\text{C},p\big) - \Psi - \Phi^0\Big)}}\;. \tag{6}$$

Note again that height z is negative in the ocean. When the code is called with three arguments, the third argument is taken to dynamic height Ψ and the geopotential at zero pressure Φ^0 is taken to be zero. When the code is called with four arguments the third argument is taken to be Ψ and the fourth Φ^0. The dynamic height anomaly Ψ can be evaluated using the GSW function **gsw_geo_strf_dyn_height_CT**, noting that the reference pressure in the call to this function must be zero sea pressure.

Note that in Eqn. (5) the last term, $g(\phi,0)\left(z-\tfrac{1}{2}\gamma z^2\right)$, can be written as $z\bar{g}$ where $\bar{g}$ is the mean gravitational acceleration between $z=0$ and the height concerned. Recognizing this, the height z output from this algorithm is also equal to

$$z \;=\; -\frac{\left(\hat{h}^{48}\left(S_{SO},0°\mathrm{C},p\right)-\Psi-\Phi^0\right)}{\bar{g}}. \qquad (7)$$

Notes on the GSW code

gsw_p_from_z

for calculating pressure *p* from height *z*

In the **gsw_p_from_z** code we evaluate pressure p using a modified Newton-Raphson iteration procedure so that the pressure so obtained is exactly consistent with the "forward" calculation of z from p via the function **gsw_z_from_p**.

When the **gsw_p_from_z** code is called with two arguments, as in **gsw_p_from_z**(z,lat), we ignore $\Psi+\Phi^0$ while solving Eqn. (8) below. Note again that height z is negative in the ocean. When the code is called with three arguments, the third argument is taken to dynamic height Ψ and the geopotential at zero pressure Φ^0 is taken to be zero. When the code is called with four arguments the third argument is taken to be Ψ and the fourth Φ^0. The dynamic height anomaly Ψ can be evaluated using the GSW function **gsw_geo_strf_dyn_height_CT**, noting that the reference pressure in the call to this function must be zero sea pressure.

A good starting point for pressure is found by using the Saunders (1981) quadratic expression relating depth to a quadratic of pressure; we solve this quadratic using the standard quadratic solution formula but for p^{-1} instead of for p, so that the solution is well-behaved as z goes to zero.

Hence, given z, we have a zeroth estimate of pressure, p_0, from the Saunders (1981) quadratic expression. Now we want to solve (see Eqn. (3.32.3) of the TEOS-10 Manual, IOC *et al*. (2010)),

$$f(p)=0, \quad \text{where} \quad f(p) \;=\; \hat{h}^{48}\left(S_{SO},0°\mathrm{C},p\right)-\Psi-\Phi^0+g(\phi,0)\left(z-\tfrac{1}{2}\gamma z^2\right). \qquad (8)$$

The derivative of $f(p)$ is approximately

$$f'(p) \;=\; 10^4\,\hat{v}^{48}\left(S_{SO},0°\mathrm{C},p\right), \qquad (9)$$

and this is available from the 48-term rational function expression for seawater density (and since $\Theta=0°\mathrm{C}$, $\hat{v}^{48}\left(S_{SO},0°\mathrm{C},p\right)$ is particularly simple to evaluate using the library function **gsw_specvol_SSO_0_p**(p)). The factor of 10^4 in Eqn. (9) is because we want to solve for pressure in dbar rather than in the natural SI unit for pressure of Pa. That is, Eqn. (9) is the derivative of $f(p)$ with respect to pressure p in dbar.

After finding p_0 we evaluate $f(p_0) = \hat{h}^{48}(S_{SO}, 0°C, p_0) - \Psi - \Phi^0 + g(\phi,0)\left(z - \tfrac{1}{2}\gamma z^2\right)$, then calculate $f'(p_0) = 10^4\,\hat{v}^{48}(S_{SO}, 0°C, p_0)$ and use these values of $f(p_0)$ and $f'(p_0)$ to form an intermediate pressure estimate p_1 as (this is a standard Newton's method iteration)

$$p_1 = p_0 - f(p_0)/f'(p_0) \; . \tag{8}$$

Then we form $p_m = 0.5(p_0 + p_1)$ and evaluate $f'(p_m) = 10^4\,\hat{v}^{48}(S_{SO}, 0°C, p_m)$ and use $f(p_0)$ and $f'(p_m)$ to calculate p_2 from

$$p_2 = p_0 - f(p_0)/f'(p_m) \; . \tag{9}$$

This is one full step of the "modified Newton-Raphson" iteration procedure of McDougall and Wotherspoon (2013), and this one modified step gives pressure to better than $1.6 x 10^{-10}$ dbar (which is essentially machine precision) down to a height z of -8000m. The **gsw_p_from_z** function performs this one full iteration of the modified Newton-Raphson iteration.

References

McDougall T. J. and S. J. Wotherspoon, 2013: A simple modification of Newton's method to achieve convergence of order $1+\sqrt{2}$. Manuscript

Saunders, P. M, 1981: Practical conversion of pressure to depth. *Journal of Physical Oceanography*, **11**, 573-574.

Below is section 3.32 of the TEOS-10 Manual (IOC *et al*. (2010)).

3.32 Pressure to height conversion

The vertical integral of the hydrostatic equation ($g = -v P_z$) can be written as

$$\int_0^z g(z')\,dz' = \Phi^0 - \int_{P_0}^{P} v(p')\,dP' = -\int_{P_0}^{P} \hat{v}(S_{SO}, 0°C, p')\,dP' + \Psi + \Phi^0$$
$$= -\,\hat{h}(S_{SO}, 0°C, p) + \Psi + \Phi^0, \tag{3.32.1}$$

where the dynamic height anomaly Ψ is expressed in terms of the specific volume anomaly $\hat{\delta} = \hat{v}(S_A, \Theta, p) - \hat{v}(S_{SO}, 0°C, p)$ by

$$\Psi = -\int_{P_0}^{P} \hat{\delta}(p')\,dP', \tag{3.32.2}$$

where $P_0 = 101\,325\,\text{Pa}$ is the standard atmosphere pressure. Writing the gravitational acceleration of Eqn. (D.3) as $g = g(\phi, z) = g(\phi,0)(1 - \gamma z)$, the left-hand side of Eqn. (3.32.1) becomes $g(\phi,0)\left(z - \tfrac{1}{2}\gamma z^2\right)$, and using the 48-term expression for the specific enthalpy of Standard Seawater, Eqn. (3.32.1) becomes

$$\hat{h}^{48}(S_{SO}, 0°C, p) - \Psi - \Phi^0 + g(\phi,0)\left(z - \tfrac{1}{2}\gamma z^2\right) = 0 \; . \tag{3.32.3}$$

This is the equation that is solved for height z in the GSW function **gsw_z_from_p**. It is traditional to ignore $\Psi + \Phi^0$ when converting between pressure and height, and this can be done by simply calling this function with only two arguments, as in **gsw_z_from_p**(p,lat). Ignoring $\Psi + \Phi^0$ makes a difference to z of up to 4m at 5000 dbar. Note that height z is negative in the ocean. When the code is called with three arguments, the third argument is taken to be $\Psi + \Phi^0$ and this is used in the solution of Eqn. (3.32.3). Dynamic height anomaly Ψ can be evaluated using the GSW function **gsw_geo_strf_dyn_height**. The GSW function **gsw_p_from_z** is the exact inverse function of **gsw_z_from_p**; these functions yield outputs that are consistent with each other to machine precision.

When vertically integrating the hydrostatic equation $P_z = -g\rho$ in the context of an ocean model where Absolute Salinity S_{A} and Conservative Temperature Θ are piecewise constant in the vertical, the geopotential (Eqn. (3.24.2))

$$\Phi = \int_0^z g(z')\, dz' = \Phi^0 - \int_{P_0}^{P} v(p')\, dP', \tag{3.32.4}$$

can be evaluated as a series of exact differences. If there are a series of layers of index i separated by pressures p^i and p^{i+1} (with $p^{i+1} > p^i$) then the integral can be expressed (making use of (3.7.5), namely $h_P\big|_{S_{\mathrm{A}},\Theta} = \hat{h}_P = v$) as a sum over n layers of the differences in specific enthalpy so that

$$\Phi = \Phi^0 - \int_{P_0}^{P} v(p')\, dP' = \Phi^0 - \sum_{i=1}^{n}\Big[\hat{h}\big(S_{\mathrm{A}}^i,\Theta^i,p^{i+1}\big) - \hat{h}\big(S_{\mathrm{A}}^i,\Theta^i,p^i\big)\Big]. \tag{3.32.5}$$

The difference in enthalpy at two different pressures for given values of S_{A} and Θ is available in the GSW Oceanographic Toolbox via the function **gsw_enthalpy_diff**. The summation of a series of such differences in enthalpy occurs in the GSW functions to evaluate two geostrophic streamfunctions from piecewise-constant vertical property profiles, **gsw_geo_strf_dyn_height_pc** and **gsw_geo_strf_isopycnal_pc**.