Notes on the function $\text{gsw\_pt\_first\_derivatives}(S_A, CT)$

This function, $\text{gsw\_pt\_first\_derivatives}(S_A, CT)$, evaluates the first derivatives of potential temperature with respect to Conservative Temperature and Absolute Salinity, as given in Eqn. (A.12.6) in the TEOS-10 Manual (IOC et al. (2010)), repeated here.

$$\left. \frac{\partial \theta}{\partial S_A} \right|_{\theta_0} = \frac{c_p^0}{c_p}(S_A, \theta_0),$$

$$\left. \theta_{S_A} \right|_{\theta} = \frac{\partial S_A}{\partial \theta} = -\left[ \mu(S_A, \theta_0) - \left( T_0 + \theta_0 \right) \mu_T(S_A, \theta_0) \right] \frac{1}{c_p}(S_A, \theta_0), \quad (A.12.6a,b)$$

This function $\text{gsw\_pt\_first\_derivatives}(S_A, CT)$ achieves this by calling another GSW function, $\text{gsw\_CT\_first\_derivatives}(S_A, pt)$, because, from Eqn. (A.12.9a,b) (repeated below) the first derivatives $\frac{\partial \theta}{\partial \theta}$ and $\frac{\partial S_A}{\partial \theta}$ are simply related to $\frac{\partial \theta}{\partial S_A}$ and $\frac{\partial S_A}{\partial \theta}$.

$$\frac{\partial \theta}{\partial \theta} = \frac{1}{\theta}, \quad \frac{\partial S_A}{\partial \theta} = -\frac{\partial S_A}{\partial \theta}, \quad \frac{\partial \theta}{\partial \theta} = -\frac{\partial S_A}{\partial \theta} - \frac{\partial \theta}{\partial \theta} + \frac{\partial S_A}{\partial \theta} \frac{\partial \theta}{\partial \theta}, \quad (A.12.9a,b,c,d)$$

The code evaluates potential temperature and then calls $\text{gsw\_CT\_first\_derivatives}(S_A, pt)$ to find $\frac{\partial \theta}{\partial \theta}$ and $\frac{\partial S_A}{\partial \theta}$, and then these are used to find $\frac{\partial \theta}{\partial S_A}$ and $\frac{\partial S_A}{\partial \theta}$.

Hence, this function $\text{gsw\_pt\_first\_derivatives}(S_A, CT)$ is essentially the following four lines of code.

```python
pt = gsw.pt_from_CT(SA, CT);
[CT_SA, CT_pt] = gsw_CT_first_derivatives(SA, pt);
pt_CT = ones(size(CT_pt))./CT_pt;
pt_SA = -CT_SA.*pt_CT;
```

This function is well behaved at $S_A = 0 \, \text{g kg}^{-1}$.

References


Here follows appendix A.12 of the TEOS-10 Manual (IOC et al., 2010).
A.12 Differential relationships between \( \eta, \theta, \Theta \) and \( S_\lambda \)

Evaluating the fundamental thermodynamic relation in the forms (A.11.6) and (A.11.12) and using the four boxed equations in appendix A.11, we find the relations

\[
(T_0 + \tau) d\eta + \mu(p) dS_\lambda = \frac{(T_0 + \tau) c_p(0) d\theta}{(T_0 + \theta)} + \mu(p) - \mu(0) dS_\lambda. 
\]

(A.12.1)

The quantity \( \mu(p) dS_\lambda \) is now subtracted from each of these three expressions and the whole equation is then multiplied by \( (T_0 + \theta)/(T_0 + \tau) \) obtaining

\[
(T_0 + \theta) d\eta = c_p(0) d\theta - (T_0 + \theta) \mu_T(0) dS_\lambda = c_p^0 d\Theta - \mu(0) dS_\lambda. 
\]

(A.12.2)

From this follows all the following partial derivatives between \( \eta, \theta, \Theta \) and \( S_\lambda \),

\[
\Theta_{\eta}|_{S_\lambda} = c_p(S_\lambda, \theta, 0)/c_p^0, \quad \Theta_{S_\lambda}|_{\theta} = \frac{\mu(S_\lambda, \theta, 0) - (T_0 + \theta) \mu_T(S_\lambda, \theta, 0)}{c_p^0}, 
\]

(A.12.3)

\[
\Theta_{\theta}|_{S_\lambda} = \frac{(T_0 + \theta)}{c_p^0}, \quad \Theta_{S_\lambda}|_{\eta} = \frac{\mu(S_\lambda, \theta, 0)}{c_p^0}, 
\]

(A.12.4)

\[
\Theta_{\Theta}|_{S_\lambda} = \frac{(T_0 + \theta)}{c_p(S_\lambda, \theta, 0)}, \quad \Theta_{S_\lambda}|_{\Theta} = \frac{(T_0 + \theta) \mu_T(S_\lambda, \theta, 0)}{c_p(S_\lambda, \theta, 0)}, 
\]

(A.12.5)

\[
\Theta_{\eta}|_{S_\lambda} = c_p^0/c_p(S_\lambda, \theta, 0), \quad \Theta_{S_\lambda}|_{\eta} = -\frac{\mu(S_\lambda, \theta, 0) - (T_0 + \theta) \mu_T(S_\lambda, \theta, 0)}{c_p(S_\lambda, \theta, 0)}, 
\]

(A.12.6)

\[
\Theta_{\theta}|_{S_\lambda} = c_p(S_\lambda, \theta, 0)/(T_0 + \theta), \quad \Theta_{S_\lambda}|_{\Theta} = -\mu_T(S_\lambda, \theta, 0), 
\]

(A.12.7)

\[
\Theta_{\Theta}|_{S_\lambda} = -\mu_T(S_\lambda, \theta, 0)/(T_0 + \theta). 
\]

(A.12.8)

The three second order derivatives of \( \hat{\eta}(S_\lambda, \Theta) \) are listed in Eqns. (P.14) and (P.15) of appendix P. The corresponding derivatives of \( \hat{\theta}(S_\lambda, \Theta) \), namely \( \hat{\theta}_\eta, \hat{\theta}_S, \hat{\theta}_{\Theta S}, \hat{\theta}_{S \Theta}, \hat{\theta}_{S \Theta S} \) and \( \hat{\theta}_{S S S} \), can also be derived using Eqn. (P.13), obtaining

\[
\hat{\theta}_\eta = \frac{1}{\Theta_{\eta}}, \quad \hat{\theta}_S = -\frac{\Theta_{SS}}{\Theta_{\eta}}, \quad \hat{\theta}_{\Theta S} = -\frac{\Theta_{\Theta S}}{\Theta_{\eta}}, \quad \hat{\theta}_{S \Theta} = -\frac{\Theta_{S \Theta}}{\Theta_{\eta}}, \quad \hat{\theta}_{S S} = -\frac{\Theta_{S S}}{\Theta_{\eta}}, \quad \hat{\theta}_{S S S} = -\frac{\Theta_{S S S}}{\Theta_{\eta}}, \quad \hat{\theta}_{S S S} = -\frac{\Theta_{S S S}}{\Theta_{\eta}}, 
\]

(A.12.9a,b,c,d)

and

\[
\hat{\theta}_{S S S} = -\frac{\Theta_{S S S}}{\Theta_{\eta}} + 2 \frac{\Theta_{S \Theta}}{\Theta_{\eta}} \frac{\Theta_{S \Theta}}{\Theta_{\eta}} - \left( \frac{\Theta_{S S}}{\Theta_{\eta}} \right)^2 \frac{\Theta_{\Theta}}{\Theta_{\eta}}, 
\]

(A.12.10)

in terms of the partial derivatives \( \Theta_{\theta}, \Theta_{S}, \Theta_{\Theta S}, \Theta_{\Theta S}, \Theta_{S S} \) and \( \Theta_{S S S} \), which can be obtained by differentiating the polynomial \( \hat{\Theta}(S_\lambda, \Theta) \) from the TEOS-10 Gibbs function.