DERIVING THE VERTICAL VARIATION OF THE GRAVITATIONAL ACCELERATION IN THE OCEAN

Notes written by Trevor McDougall, 27th July 2010

Fact A. Total mass of the earth (including its oceans) is

$$M = 5.9742 \text{ x } 10^{24} \text{ kg} . \qquad (1)$$

Fact B. Radius of the earth is (radius of the sphere having the same volume as the earth, and this is the radius that MOM and other GFDL models, ice, atmosphere, use)

$$a = 6.371 \text{ x } 10^{6} \text{ m} . \qquad (2)$$

Fact C. Volume of the earth ($\frac{4}{3}\pi a^3$) is

$$V = 1.08321 \text{ x } 10^{21} \text{ m}^3 . \qquad (3)$$

Fact D. The gravitational acceleration inside a sphere of uniform density varies linearly with radius from the centre of the sphere to the outside.

Fact E. The gravitational acceleration outside of a sphere decreases as the reciprocal of the square of the distance from the centre of the sphere. That is, as far as this external gravitational field is concerned, the mass of the sphere behaves as though it is concentrated at the centre of the sphere. This is true so long as the density of material in the sphere is a function of radius only, and not also a function of latitude and longitude.

Assumption 1: That the average density of seawater is 1035 kg m^{-3} (from Gill (1982) and Griffies (2004)).

> I think that the above 5 facts and 1 assumption are enough to quantify the vertical variation of the gravitational acceleration inside the ocean. What I get below is almost identical to what Saunders and Fofonoff (1976) and Saunders (1981) quote without a reference. An email from Bob Hallberg in mid 2009 pointed out Fact D above, and this is key. In fact, I think that Bob essentially derived what I do below.

The conceptual Seawater Sphere and point mass "solid" earth

Imagine that planet earth is in fact the sum of a sphere of seawater, plus a single point mass at the centre of the earth. The sum of the mass of the seawater sphere and the point mass, is set equal to the total mass of the earth M of Eqn. (1).

The mass of the seawater sphere is

$$\textit{mass of Seawater Sphere} = 1035 \times 1.08321 \times 10^{21} \text{ kg} = 1.1211 \times 10^{24} \text{ kg}. \tag{4}$$

The point mass of solid earth at the earth's centre has a mass of

$$\textit{point mass at the earth's centre} = (5.9742 - 1.1211) \times 10^{24} \text{ kg} . \tag{5}$$

The ratio R of the mass of the seawater sphere to the total mass of the earth is then

$$R = \frac{1.1211}{5.9742} = 0.187661 . \tag{6}$$

Split the gravitational acceleration into the sum of the two parts

So, at the earth's surface we can regard the fraction R of the gravitational acceleration g as being due to the attraction of the seawater sphere (which occupies the whole volume of the earth) plus the fraction $(1 - R)$ of the gravitational acceleration g due to the attraction of the point source of solid mass located at the centre of the earth (which occupies zero volume, although this zero volume aspect is not essential).

Recognize that we do know how the gravitational acceleration of both parts vary with depth in the ocean

The beauty of this particular decomposition of the earth's mass is that as we go down into the ocean, we do know how the gravitational acceleration of both the two individual components vary with depth. For the seawater sphere, its gravitational acceleration, which is $R\,g(\phi,0)$ at the sea surface, decreases linearly with depth, to become zero at the earth's centre. (Here $g(\phi,0)$ stands for the full gravitational acceleration at latitude ϕ and at zero pressure, from Moritz (2000)). For the point mass at the centre of the earth, its contribution to the gravitational acceleration increases in proportion to the inverse square of the radial distance to the centre of the earth.

So long as we are considering depths that are actually in the ocean and not in the solid earth below the ocean, this decomposition into a seawater sphere and a point mass at the earth centre is legitimate.

Doing the sums

Take z to be positive upwards, so it is negative in the ocean, starting at $z = 0$ at the geoid. The above considerations imply that

$$\frac{g(\phi,z)}{g(\phi,0)} = R(1+z/a) + (1-R)(1+z/a)^{-2}, \qquad \text{ocean} \qquad (7)$$

and by contrast, in the atmosphere we have

$$\frac{g(\phi,z)}{g(\phi,0)} = (1+z/a)^{-2}. \qquad \text{atmosphere} \qquad (8)$$

A Taylor series expansion of Eqn. (7) in terms of the height z from the sea surface gives

$$g(\phi,z) = g(\phi,0)\left[1-(2-3R)\left(\frac{z}{a}\right)+3(1-R)\left(\frac{z}{a}\right)^2\right]+\ldots. \qquad \text{ocean} \qquad (9)$$

The quadratic term is small and can be ignored, so that

$$g(\phi,z) \approx g(\phi,0)\left[1-(2-3R)\left(\frac{z}{a}\right)\right], \qquad \text{ocean} \qquad (10)$$

whereas the corresponding expression for the atmosphere (for small z) is

$$g(\phi,z) \approx g(\phi,0)\left[1-2\left(\frac{z}{a}\right)\right]. \qquad \text{atmosphere} \qquad (11)$$

The ratio $(2-3R)/2$, using R from Eqn. (6), is 0.7185 so that the gravitational acceleration increases with depth in the ocean at 71.85% of the rate at which the gravitational acceleration decreases with height in the atmosphere.

Using the values of a and R from Eqns. (2) and (6), Eqn. (10) becomes

$$g(\phi,z) \approx g(\phi,0)\left(1-2.2556x10^{-7}\, z/(\mathrm{m})\right), \qquad \text{ocean} \qquad (12)$$

and this agrees very closely with the corresponding expression $g(\phi,z) = g(\phi,0) - 2.226x10^{-6}\, z/(\mathrm{m})$ which is used without attribution by both Saunders and Fofonoff (1976) and by Saunders (1981).

I suggest that we round off the number in (12) a little to take the following as "gospel" (and define $\gamma = 2.26x10^{-7}\,\mathrm{m}^{-1}$)

$$g(\phi,z) = g(\phi,0)\left(1-2.26x10^{-7}\, z/(\mathrm{m})\right), \qquad \text{ocean} \qquad (12a)$$

An approximate expression in terms of pressure in the ocean rather than z is found by using $p/(\mathrm{dbar}) = -9.7963x0.1035\, z/(\mathrm{m})$ in Eqn. (12) obtaining

$$g(\phi,z) \approx g(\phi,0)\left(1+2.22x10^{-7}\, p/(\mathrm{dbar})\right). \qquad \text{ocean} \qquad (13)$$

Equations (12a) and (13) are what has been adopted by TEOS-10 and appears in Appendix D of the TEOS-10 Manual (IOC *et. al.*, 2010).

References

IOC, SCOR and IAPSO, 2010: *The international thermodynamic equation of seawater – 2010: Calculation and use of thermodynamic properties*. Intergovernmental Oceanographic Commission, Manuals and Guides No. 56, UNESCO (English), 196 pp.

Saunders, P. M, 1981: Practical conversion of pressure to depth. *Journal of Physical Oceanography*, **11**, 573-574.

Saunders, P. M. and N. P. Fofonoff, 1976: Conversion of pressure to depth in the ocean. *Deep-Sea Research*, **23**, 109-111.

Here follows the part of Appendix D of IOC et al. (2010) that is concerned with the gravitational acceleration.

Gravitational Acceleration

The gravitational acceleration g in the ocean can be taken to be the following function of latitude ϕ and sea pressure p, or height z relative to the geoid,

$$\begin{aligned} g/(\mathrm{m\ s^{-2}}) &= 9.780\,327\left(1 + 5.3024\times10^{-3}\sin^2\phi - 5.8\times10^{-6}\sin^2 2\phi\right)\left(1 - 2.26\times10^{-7}\,z/(\mathrm{m})\right) \\ &= 9.780\,327\left(1 + 5.2792\times10^{-3}\sin^2\phi + 2.32\times10^{-5}\sin^4\phi\right)\left(1 - 2.26\times10^{-7}\,z/(\mathrm{m})\right) \\ &\approx 9.780\,327\left(1 + 5.2792\times10^{-3}\sin^2\phi + 2.32\times10^{-5}\sin^4\phi\right)\left(1 + 2.22\times10^{-7}\,p/(\mathrm{dbar})\right). \end{aligned} \tag{D.3}$$

The dependence on latitude in Eqn. (D.3) is from Moritz (2000) and is the gravitational acceleration on the surface of an ellipsoid which approximates the geoid. The variation of g with z and p in the ocean in Eqn. (D.3) is derived in McDougall *et al*. (2013). The height z above the geoid is negative in the ocean. Note that g increases with depth in the ocean at about 71.85% of the rate at which it decreases with height in the atmosphere.

At a latitude of $45°\mathrm{N}$ and at $p = 0$, $g = 9.8062\ \mathrm{m\ s^{-2}}$, which is a value commonly used in ocean models. The value of g averaged over the earth's surface is $g = 9.7976\ \mathrm{m\ s^{-2}}$, while the value averaged over the surface of the ocean is $g = 9.7963\ \mathrm{m\ s^{-2}}$ (Griffies (2004)).