Notes on the function
gsw_geo_strf_isopycnal_pc(SA,CT,delta_p,Neutral_Density,layer_index,A)

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In density-coordinate forward ocean models, the variables are usually interpreted as being piecewise constant in the vertical, and this function, gsw_geo_strf_isopycnal_pc, calculates the approximate “isopycnal” geostrophic streamfunction of McDougall and Klocker (2010) taking this piecewise-constant nature of Absolute Salinity and Conservative Temperature into account. The code uses the 75-term polynomial function expression for specific volume gsw_specvol(SA,CT,p). This 75-term polynomial expression for specific volume is discussed in Roquet et al. (2015) and in appendix A.30 and appendix K of the TEOS-10 Manual (IOC et al. (2010)). For dynamical oceanography we may take the 75-term polynomial expression for specific volume as essentially reflecting the full accuracy of TEOS-10.

In density-coordinate models (often called “isopycnal models”) the “thickness” of the \( i \)th layer, \( \rho^{i+1} - \rho^i \), varies with latitude, longitude and time. In these layered models the “thicknesses” of the least dense layers are often zero at many locations in the ocean, because these density surfaces have outcropped.

The McDougall-Klocker geostrophic streamfunction is defined by

\[
\varphi^a = \frac{1}{2} \left( P - \bar{P} \right) \tilde{\delta} \left( S_A, \Theta, p \right) - \frac{1}{2} \rho^a \bar{v} \left( \Theta - \bar{\Theta} \right) \left( P - \bar{P} \right)^2 - \int_{\bar{h}}^{P} \tilde{\delta} \, dP',
\]

(1)

where \( \rho^{-1} \bar{v} \) is taken to be the constant value \( 2.7 \times 10^{-15} \text{K}^{-1} \text{Pa}^{-2} \text{s}^{-2} \). This geostrophic streamfunction is for use in approximately neutral surfaces, such as an \( \omega \)-surface of Klocker et al. (2009a,b) or a Neutral Density (\( \gamma^a \)) surface of Jackett and McDougall (1997) or a suitably referenced potential density surface such as a \( \sigma_\omega \) surface. A suitable reference seawater parcel \( \left( \tilde{S}_A, \tilde{\Theta}, \tilde{P} \right) \) is selected for each approximately neutral surface that one is considering, and the specific volume anomaly \( \tilde{\delta} \) is defined as

\[
\tilde{\delta} \left( S_A, \Theta, p \right) = \tilde{v} \left( S_A, \Theta, p \right) - \tilde{v} \left( \tilde{S}_A, \tilde{\Theta}, \tilde{P} \right).
\]

(2)

In this function, gsw_geo_strf_isopycnal_pc, the reference values \( \left( \tilde{S}_A, \tilde{\Theta}, \tilde{P} \right) \) are found by interpolation down a single reference cast which includes these three variables as well as values of Neutral Density \( \gamma^a \) and also of \( \sigma_\omega \). This reference cast is interpolated with respect to \( \gamma^a \), to the value of \( \gamma^a \) which is supplied by the user of this function, with one value of \( \gamma^a \) for each “density” surface that is selected. If this function is being used with a series of approximately neutral surfaces which are neither Neutral Density surfaces nor \( \omega \)-surfaces, then a value of \( \gamma^a \) needs to be assigned to each of the user’s surfaces. This could be achieved, for example, by the user labeling a vertical profile of her/his data from the mid equatorial Pacific with Neutral Density and associating these values of Neutral Density to the whole of the user’s layers; with one value of \( \gamma^a \) for each selected approximately neutral surface. As an alternative to having to label each layer with Neutral Density, we have also allowed for the option of having the reference data set be interpolated with respect to \( \sigma_\omega \), the potential density with respect to 2000 dbar. In this case, the user needs to supply the \( \sigma_\omega \) value for each layer on which the McDougall-Klocker streamfunction is required.

The last term in Eqn. (1) is obtained with the aid of a call to gsw_geo_strf_dyn_height_pc which returns the dynamic height anomaly \( \Psi \) for the piecewise constant vertical profile, with respect to \( S_{SO} = 35.165 \text{ g kg}^{-1} \) and \( \Theta = 0^\circ \text{C} \). The last term in Eqn. (1) becomes
\[- \frac{\partial}{\partial \bar{h}} \tilde{\phi} \delta P' = \Psi + \hat{h}\left(\tilde{S}_A, \tilde{\Theta}, \hat{\varnothing} \left[ \frac{1}{2} p^{n+1} + p^n \right] \right) - c_p \hat{\varnothing} - \hat{h}\left(S_{SO}, \Theta = 0^\circ C, \frac{1}{2} \left[ p^{n+1} + p^n \right] \right) \]  

(3)

where \( \frac{1}{2} \left[ p^{n+1} + p^n \right] \) is the pressure at the mid-pressure of the \( n \)th layer. The second term in Eqn. (3) is obtained from a call to \textit{gsw_enthalpy} and the last term via a call to \textit{gsw_enthalpy}. Combining Eqs. (1) and (3), the expression that is used to evaluate \textit{gsw_geo_strf_isopycnal_pc} is

\[
\phi^n = \frac{1}{2} \left( P - \tilde{P} \right) \tilde{\delta} \left( S_A, \Theta, p \right) - \frac{1}{2} \rho^{-1} \tilde{T}_n \left( \Theta - \tilde{\Theta} \right) \left( P - \tilde{P} \right)^2 + \Psi
\]

\[
+ \hat{h}\left( \tilde{S}_A, \tilde{\Theta}, \frac{1}{2} \left[ p^{n+1} + p^n \right] \right) - c_p \hat{\varnothing} - \hat{h}\left(S_{SO}, \Theta = 0^\circ C, \frac{1}{2} \left[ p^{n+1} + p^n \right] \right),
\]

(4)

where \( \Psi \) is the output of \textit{gsw_geo_strf_dyn_height_pc}.

The pressure input to the function \textit{gsw_geo_strf_isopycnal_pc} is the matrix \( \text{delta}_p = p^{i+1} - p^i \) and the code can handle zero values of this layer “thickness” in several layers (which occurs when density surface outcrop). The function calculates the pressures \( p_{\text{mid}} \) at the mid-pressure of each layer by summing the \( \text{delta}_p \) of the shallower layers and adding half of \( \text{delta}_p \) of the shallower layers and adding half of \( \text{delta}_p \) to \( p^i \) at the \( i \)th layer (this summation is performed by \textit{gsw_geo_strf_dyn_height_pc} and returned to the present function).

In \textit{gsw_geo_strf_isopycnal_pc} \((\text{SA},\text{CT},\text{delta}_p,\text{Neutral_Density},\text{layer_index},A)\) the user supplies the Absolute Salinity \( \text{SA} \), conservative Temperature \( \text{CT} \) and pressure thickness \( \text{delta}_p \) of many layers, with the first such layer being bounded by the sea surface. The user can request the McDougall-Klocker geostrophic streamfunction on each layer, or he/she can specify the indices of the layers on which the streamfunction is requested in the vector \( A \). For example, \( A = [6;12] \) implies that the user wants the streamfunction on layers 6 to 12 inclusive, and the vector \text{Neutral_Density} must be consistent with this choice, so that in this case, \text{Neutral_Density} must be 7 long. The last argument of the calling sequence to this function must be either ‘\text{gn}’ or ‘\text{s2}’ which tells the code whether the interpolating variable for the reference cast is \( \gamma^n \) or \( \sigma_2 \).

The two-dimensional gradient of \( \phi^n \) of Eqn. (4) in an approximately neutral surface \( \text{ans} \) is simply related to the difference between the horizontal geostrophic velocity \( \mathbf{v} \) in this surface and that at the sea surface, \( \mathbf{v}_0 \), according to

\[
k \times \nabla_{\text{ans}} \phi^n = f\mathbf{v} - f\mathbf{v}_0 \quad \text{or} \quad \nabla_{\text{ans}} \phi^n = -k \times \left( f\mathbf{v} - f\mathbf{v}_0 \right).
\]

(5)

Note that in this function, the reference pressure of the reference surface here is the sea surface. This is because with piecewise-constant vertical profiles of Absolute Salinity and Conservative Temperature and with variable pressure thicknesses of the layers, it is not possible to evaluate a geostrophic streamfunction at a fixed pressure.

**A remark on the use of the isopycnal geostrophic streamfunction of McDougall and Klocker (2010) in layered ocean models**

Section 8 of McDougall and Klocker (2010) discusses the use of the geostrophic streamfunction given by Eqn. (1) in density-coordinate ocean models. Here we expand a little on this discussion.

If the flow in the ocean is geostrophic then the difference between the horizontal velocity at pressure \( P \) and that at the sea surface is given by

\[
-k \times (f\mathbf{v} - f\mathbf{v}_0) = \left\{ \frac{1}{\nu} \nabla^2 P - \nabla \Phi_0 \right\},
\]

(4)
where $\Phi_0$ is the geopotential at the sea surface, and the use of upper case $P$ for pressure serves to remind us that it needs to be in Pa in these expressions. From Eqn. (6) of McDougall and Klocker (2010) we see that \( \frac{1}{\rho} \nabla_z P \cdot \nabla \Phi_0 \) can be expressed in terms of the two-dimensional gradient of $\varphi^n$ (of Eqn. (1) above) in any general surface by

\[
\frac{1}{\rho} \nabla_z P \cdot \nabla \Phi_0 = \nabla_{\sigma} \left[ \varphi^n + \frac{1}{12} \rho^{-1} T_b^{\varphi} \left( \Theta - \tilde{\Theta} \right) \left( P - \tilde{P} \right)^2 \right] - \frac{1}{2} \frac{1}{\rho} \left( P - \tilde{P} \right)^2 \nabla_{\sigma} \delta + \frac{1}{2} \delta \nabla_{\sigma} P. \tag{5}
\]

This equation demonstrates that in this application to a forward ocean model where all terms on the right-hand side of Eqn. (5) are evaluated during a model run, there is no point including the

\[
- \frac{1}{12} \rho^{-1} T_b^{\varphi} \left( \Theta - \tilde{\Theta} \right) \left( P - \tilde{P} \right)^2 \tag{6}
\]

term in the calculation of the geostrophic streamfunction (1) since it is subtracted from this streamfunction when it is used in Eqn. (5) in the forward model.

Hence, if this suggestion were adopted in a layered ocean model, the geostrophic streamfunction would be

\[
\varphi = \frac{1}{2} \left( P - \tilde{P} \right) \delta \left( S_p, \Theta, p \right) - \int_{b}^{a} \delta \ dp \tag{7}
\]

and the desired horizontal pressure gradient term would be calculated in the forward ocean model using the right-hand side of the following equation,

\[
\frac{1}{\rho} \nabla_z P \cdot \nabla \Phi_0 = \nabla_{\sigma} \varphi^n - \frac{1}{2} \left( P - \tilde{P} \right) \nabla_{\sigma} \delta + \frac{1}{2} \delta \nabla_{\sigma} P. \tag{8}
\]

References


Here follows section 3.30 from the TEOS-10 Manual (IOC et al. (2010)).

### 3.30 Geostrophic streamfunction in an approximately neutral surface

In order to evaluate a relatively accurate expression for the geostrophic streamfunction in an approximately neutral surface a suitable reference seawater parcel \( \{ \widetilde{S}_A, \widetilde{\Theta}, \widetilde{P} \} \) is selected from the approximately neutral surface that one is considering, and the specific volume anomaly \( \tilde{\delta} \) is defined as in (3.7.3) above. The approximate geostrophic streamfunction \( \varphi^a \) is given by (from McDougall and Klocker (2010))

\[
\varphi^a = \frac{1}{2} \left( P - \widetilde{P} \right) \tilde{\delta} \left( S_A, \Theta, p \right) - \frac{1}{12} \rho^{-1} T_b \left( \Theta - \widetilde{\Theta} \right) \left( P - \widetilde{P} \right)^2 - \int_{\tilde{P}}^P \tilde{\delta} \ dP'.
\] (3.30.1)

This expression is more accurate than the Montgomery and Cunningham geostrophic streamfunctions when used in potential density surfaces, in the \( \omega \)-surfaces of Klocker et al. (2009a,b) and in the Neutral Density surfaces of Jackett and McDougall (1997). That is, in these surfaces \( \nabla \varphi^a \approx \frac{1}{\rho} \nabla P - \nabla \Phi_0 = -k \times (\nabla - \nabla \Phi_0) \) to a very good approximation. In Eqn. (3.30.1) \( \rho^{-1} T_b \) is taken to be the constant value \( 2.7 \times 10^{-15} \text{K}^{-1} \text{(Pa)}^{-2} \text{m}^2\text{s}^{-2} \). This approximate isopycnal geostrophic streamfunction of McDougall and Klocker (2010) is available as the function `gsw_geo_strf_isopycnal` in the GSW Toolbox. When the last argument of this function, \( p_{\text{ref}} \), is other than zero, the function returns the isopycnal geostrophic streamfunction with respect to a (deep) reference sea pressure \( p_{\text{ref}} \), rather than with respect to the sea surface at \( p = 0 \text{ dbar} \) (i.e. \( P = P_0 \)) as in Eqn. (3.30.1).