Notes on the function gsw_entropy_first_derivatives(SA,CT)

This function, **gsw_entropy_first_derivatives**(SA,CT), evaluates the first order partial derivatives of entropy $\hat{\eta}(S_{\mathrm{A}},\Theta)$ with respect to Absolute Salinity and Conservative Temperature, as given in Eqns. (A.12.8a) and (A.12.8b) of the TEOS-10 Manual (IOC *et al.*, 2010), repeated here,

$$\hat{\eta}_{\Theta} = \frac{c_p^0}{(T_0+\theta)}, \qquad \text{(A.12.8a)}$$

$$\hat{\eta}_{S_{\mathrm{A}}} = -\frac{\tilde{\mu}(S_{\mathrm{A}},\theta,0)}{(T_0+\theta)} = -\frac{g_{S_{\mathrm{A}}}(S_{\mathrm{A}},\theta,0)}{(T_0+\theta)}. \qquad \text{(A.12.8b)}$$

This function uses the full TEOS-10 Gibbs function $g(S_{\mathrm{A}},t,p)$ of IOC *et al.* (2010), being the sum of the IAPWS-09 and IAPWS-08 Gibbs functions. The function first calculates potential temperature θ from which $\hat{\eta}_{\Theta}$ follows directly. The term $g_{S_{\mathrm{A}}}(S_{\mathrm{A}},\theta,0)$ on the right-hand side of Eqn. (A.12.8b) contain a logarithmic singularity in the square root of Absolute Salinity so that $\hat{\eta}_{S_{\mathrm{A}}}$ does not converge as S_{A} approaches zero.

References

IAPWS, 2008: Release on the IAPWS Formulation 2008 for the Thermodynamic Properties of Seawater. The International Association for the Properties of Water and Steam. Berlin, Germany, September 2008, available from www.iapws.org. This Release is referred to in the text as **IAPWS-08**.

IAPWS, 2009: Supplementary Release on a Computationally Efficient Thermodynamic Formulation for Liquid Water for Oceanographic Use. The International Association for the Properties of Water and Steam. Doorwerth, The Netherlands, September 2009, available from http://www.iapws.org. This Release is referred to in the text as **IAPWS-09**.

IOC, SCOR and IAPSO, 2010: *The international thermodynamic equation of seawater – 2010: Calculation and use of thermodynamic properties*. Intergovernmental Oceanographic Commission, Manuals and Guides No. 56, UNESCO (English), 196 pp. Available from http://www.TEOS-10.org

Here follows appendix A.12 and an excerpt from appendix P of the TEOS-10 Manual (IOC *et al.*, 2010).

A.12 Differential relationships between η, θ, Θ and S_{A}

Evaluating the fundamental thermodynamic relation in the forms (A.11.6) and (A.11.12) and using the four boxed equations in appendix A.11, we find the relations

$$\begin{aligned}(T_0+t)\,\mathrm{d}\eta+\mu(p)\,\mathrm{d}S_{\mathrm{A}} &= \frac{(T_0+t)}{(T_0+\theta)}\,c_p(0)\,\mathrm{d}\theta + \big[\mu(p)-(T_0+t)\,\mu_T(0)\big]\mathrm{d}S_{\mathrm{A}} \\ &= \frac{(T_0+t)}{(T_0+\theta)}\,c_p^0\,\mathrm{d}\Theta + \left[\mu(p)-\frac{(T_0+t)}{(T_0+\theta)}\mu(0)\right]\mathrm{d}S_{\mathrm{A}}\,.\end{aligned} \qquad \text{(A.12.1)}$$

The quantity $\mu(p)\mathrm{d}S_\mathrm{A}$ is now subtracted from each of these three expressions and the whole equation is then multiplied by $(T_0+\theta)/(T_0+t)$ obtaining

$$(T_0+\theta)\mathrm{d}\eta \;=\; c_p(0)\,\mathrm{d}\theta - (T_0+\theta)\,\mu_T(0)\,\mathrm{d}S_\mathrm{A} \;=\; c_p^0\,\mathrm{d}\Theta - \mu(0)\,\mathrm{d}S_\mathrm{A}\,. \qquad \text{(A.12.2)}$$

From this follows all the following partial derivatives between η, θ, Θ and S_A,

$$\Theta_\theta\big|_{S_\mathrm{A}} = c_p(S_\mathrm{A},\theta,0)/c_p^0, \qquad \Theta_{S_\mathrm{A}}\big|_\theta = \big[\mu(S_\mathrm{A},\theta,0) - (T_0+\theta)\,\mu_T(S_\mathrm{A},\theta,0)\big]/c_p^0, \qquad \text{(A.12.3)}$$

$$\Theta_\eta\big|_{S_\mathrm{A}} = (T_0+\theta)/c_p^0, \qquad \Theta_{S_\mathrm{A}}\big|_\eta = \mu(S_\mathrm{A},\theta,0)/c_p^0, \qquad \text{(A.12.4)}$$

$$\theta_\eta\big|_{S_\mathrm{A}} = (T_0+\theta)/c_p(S_\mathrm{A},\theta,0), \qquad \theta_{S_\mathrm{A}}\big|_\eta = (T_0+\theta)\,\mu_T(S_\mathrm{A},\theta,0)/c_p(S_\mathrm{A},\theta,0), \qquad \text{(A.12.5)}$$

$$\theta_\Theta\big|_{S_\mathrm{A}} = c_p^0/c_p(S_\mathrm{A},\theta,0), \quad \theta_{S_\mathrm{A}}\big|_\Theta = -\big[\mu(S_\mathrm{A},\theta,0) - (T_0+\theta)\,\mu_T(S_\mathrm{A},\theta,0)\big]/c_p(S_\mathrm{A},\theta,0), \quad \text{(A.12.6)}$$

$$\eta_\theta\big|_{S_\mathrm{A}} = c_p(S_\mathrm{A},\theta,0)/(T_0+\theta), \qquad \eta_{S_\mathrm{A}}\big|_\theta = -\mu_T(S_\mathrm{A},\theta,0), \qquad \text{(A.12.7)}$$

$$\eta_\Theta\big|_{S_\mathrm{A}} = c_p^0/(T_0+\theta), \qquad \eta_{S_\mathrm{A}}\big|_\Theta = -\mu(S_\mathrm{A},\theta,0)/(T_0+\theta). \qquad \text{(A.12.8)}$$

The three second order derivatives of $\hat{\eta}(S_\mathrm{A},\Theta)$ are listed in Eqns. (P.14) and (P.15) of appendix P. The corresponding derivatives of $\hat{\theta}(S_\mathrm{A},\Theta)$, namely $\hat{\theta}_\Theta$, $\hat{\theta}_{S_\mathrm{A}}$, $\hat{\theta}_{\Theta\Theta}$, $\hat{\theta}_{S_\mathrm{A}\Theta}$ and $\hat{\theta}_{S_\mathrm{A}S_\mathrm{A}}$ can also be derived using Eqn. (P.13), obtaining

$$\hat{\theta}_\Theta = \frac{1}{\tilde{\Theta}_\theta}, \quad \hat{\theta}_{S_\mathrm{A}} = -\frac{\tilde{\Theta}_{S_\mathrm{A}}}{\tilde{\Theta}_\theta}, \quad \hat{\theta}_{\Theta\Theta} = -\frac{\tilde{\Theta}_{\theta\theta}}{\big(\tilde{\Theta}_\theta\big)^3}, \quad \hat{\theta}_{S_\mathrm{A}\Theta} = -\frac{\tilde{\Theta}_{\theta S_\mathrm{A}}}{\big(\tilde{\Theta}_\theta\big)^2} + \frac{\tilde{\Theta}_{S_\mathrm{A}}\tilde{\Theta}_{\theta\theta}}{\big(\tilde{\Theta}_\theta\big)^3}, \qquad \text{(A.12.9a,b,c,d)}$$

$$\text{and} \quad \hat{\theta}_{S_\mathrm{A}S_\mathrm{A}} = -\frac{\tilde{\Theta}_{S_\mathrm{A}S_\mathrm{A}}}{\tilde{\Theta}_\theta} + 2\,\frac{\tilde{\Theta}_{S_\mathrm{A}}}{\tilde{\Theta}_\theta}\frac{\tilde{\Theta}_{\theta S_\mathrm{A}}}{\tilde{\Theta}_\theta} - \left(\frac{\tilde{\Theta}_{S_\mathrm{A}}}{\tilde{\Theta}_\theta}\right)^2\frac{\tilde{\Theta}_{\theta\theta}}{\tilde{\Theta}_\theta}, \qquad \text{(A.12.10)}$$

in terms of the partial derivatives $\tilde{\Theta}_\theta$, $\tilde{\Theta}_{S_\mathrm{A}}$, $\tilde{\Theta}_{\theta\theta}$, $\tilde{\Theta}_{\theta S_\mathrm{A}}$ and $\tilde{\Theta}_{S_\mathrm{A}S_\mathrm{A}}$ which can be obtained by differentiating the polynomial $\tilde{\Theta}(S_\mathrm{A},\theta)$ from the TEOS-10 Gibbs function.

And an excerpt from Appendix P of the TEOS-10 Manual (IOC *et al.*, 2010)

The partial derivatives with respect to Θ and with respect to θ, both at constant S_A and p, and the partial derivatives with respect to S_A, are related by

$$\left.\frac{\partial}{\partial\Theta}\right|_{S_\mathrm{A},p} = \frac{1}{\tilde{\Theta}_\theta}\left.\frac{\partial}{\partial\theta}\right|_{S_\mathrm{A},p}, \quad \text{and} \quad \left.\frac{\partial}{\partial S_\mathrm{A}}\right|_{\Theta,p} = \left.\frac{\partial}{\partial S_\mathrm{A}}\right|_{\theta,p} - \frac{\tilde{\Theta}_{S_\mathrm{A}}}{\tilde{\Theta}_\theta}\left.\frac{\partial}{\partial\theta}\right|_{S_\mathrm{A},p}. \qquad \text{(P.13a,b)}$$

Use of these expressions, acting on entropy yields (with $p=0$ everywhere, and using Eqn. (P.7) [or Eqn. (A.12.8b)] and Eqn. (P.8))

$$\hat{\eta}_\Theta = \frac{\tilde{\eta}_\theta}{\tilde{\Theta}_\theta} \equiv \frac{c_p^0}{(T_0+\theta)}, \qquad \hat{\eta}_{\Theta\Theta} = -\frac{1}{\tilde{\Theta}_\theta}\frac{c_p^0}{(T_0+\theta)^2}, \qquad \hat{\eta}_{S_\mathrm{A}} = -\frac{\tilde{\mu}(S_\mathrm{A},\theta,0)}{(T_0+\theta)}, \qquad \text{(P.14a,b,c)}$$

$$\hat{\eta}_{S_\mathrm{A}\Theta} = \frac{\tilde{\Theta}_{S_\mathrm{A}}}{\tilde{\Theta}_\theta}\frac{c_p^0}{(T_0+\theta)^2}, \quad \text{and} \quad \hat{\eta}_{S_\mathrm{A}S_\mathrm{A}} = -\frac{\tilde{\mu}_{S_\mathrm{A}}(S_\mathrm{A},\theta,0)}{(T_0+\theta)} - \frac{\big(\tilde{\Theta}_{S_\mathrm{A}}\big)^2}{\tilde{\Theta}_\theta}\frac{c_p^0}{(T_0+\theta)^2}, \qquad \text{(P.15a,b)}$$

in terms of the partial derivatives of the exact polynomial expressions (P.11b) and (P.12).