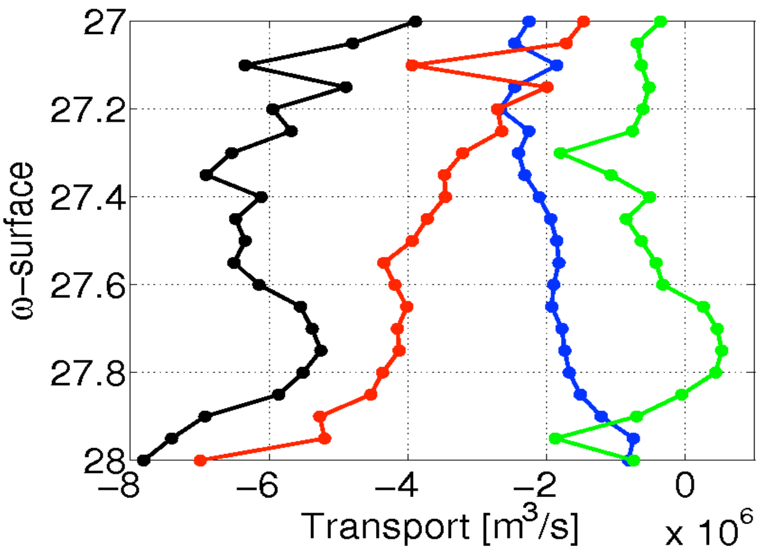


When globally integrated over complete density surfaces, the total transport due to these non-linear processes can be calculated. In green is the mean dianeutral transport from the ill-defined nature of "neutral surfaces", blue is the dianeutral transport due to cabbeling, red due to thermobaricity, and black is the total global dianeutral transport due to the sum of these three non-linear processes.

We conclude from this that while the mean dianeutral transport from the ill-defined nature of "neutral surfaces" is of leading order locally, it spatially averages to a very small transport over a complete density surface. By contrast, cabbeling and thermobaricity are predominantly downwards advection everywhere, so there is little such cancellation on area integration with these processes.

Rotation of the horizontal velocity with height

Define the angle φ (measured counter-clockwise) of the horizontal velocity $\mathbf{v}$ with respect to due east so that

$$\mathbf{v} = |\mathbf{v}|(\cos\varphi, \sin\varphi) . \qquad \text{(V\_rotate\_01)}$$

Vertically differentiate this equation and take the cross product with $\mathbf{v}$ to obtain

$$\mathbf{v}\times\mathbf{v}_z = \mathbf{k}\,\varphi_z|\mathbf{v}|^2 , \qquad \text{(V\_rotate\_02)}$$

which shows that the rate of spiraling of the horizontal velocity vector in the vertical φ_z is proportional to the amount by which this velocity is not parallel to the direction of the "thermal wind" shear $\mathbf{v}_z$. The last equation can be rewritten as

$$\varphi_z|\mathbf{v}|^2 = \mathbf{k}\cdot\mathbf{v}\times\mathbf{v}_z = uv_z - vu_z = -\,\mathbf{v}\cdot\mathbf{k}\times\mathbf{v}_z = -\,\mathbf{v}\cdot\nabla\times\mathbf{v} , \qquad \text{(V\_rotate\_03)}$$

which demonstrates that the rotation of the horizontal velocity with height is proportional to the helicity of the horizontal velocity, $\mathbf{v}\cdot\nabla\times\mathbf{v}$.

Now, substituting Eqn. (3.12.3) for the "thermal wind" $\mathbf{v}_z$, namely

$$\boxed{f\,\mathbf{v}_z = \left(\tfrac{1}{\rho}\right)_z \mathbf{k}\times\nabla_z P + \tfrac{1}{\rho}\mathbf{k}\times\nabla_z\left(P_z\right) = -\tfrac{g}{\rho}\,\mathbf{k}\times\nabla_p\rho = \tfrac{N^2}{g\rho}\,\mathbf{k}\times\nabla_n P,} \qquad (3.12.3)$$

into Eqn. (V_rotate_03) we find

$$\varphi_z|\mathbf{v}|^2 = -\,\mathbf{v}\cdot\mathbf{k}\times\mathbf{v}_z = \tfrac{N^2}{fg\rho}\,\mathbf{v}\cdot\nabla_n P . \qquad \text{(V\_rotate\_04)}$$

Under the usual Boussinesq approximation $-(g\rho)^{-1}\nabla_n P$ is set equal to the slope of the neutral tangent plane, $\nabla_n z$, so that we have

$$\varphi_z|\mathbf{v}|^2 \approx -\tfrac{N^2}{f}\,\mathbf{v}\cdot\nabla_n z , \qquad \text{(V\_rotate\_05)}$$

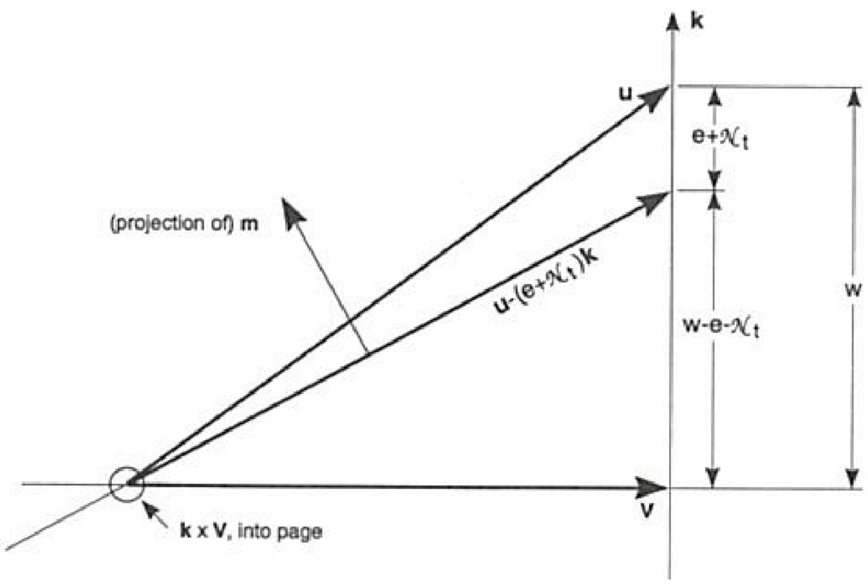
and since the vertical velocity through geopotentials, w, is given by the simple geometrical equation (where e is the vertical velocity through the neutral tangent plane),

$$w = z_t\big|_n + \mathbf{v}\cdot\nabla_n z + e , \qquad \text{(V\_rotate\_06)}$$

we have

$$\boxed{\varphi_z|\mathbf{v}|^2 \approx -\tfrac{N^2}{f}\left(w - e - z_t\big|_n\right),} \qquad \text{(V\_rotate\_07)}$$

showing that the rotation of the horizontal velocity vector with height is not simply proportional to the vertical velocity of the flow but rather only to the sliding motion along the neutral tangent plane, $\mathbf{v}\cdot\nabla_n z$.



The absolute velocity vector in the ocean

Neutral helicity is proportional to the component of the vertical shear of the geostrophic velocity ($\mathbf{v}_z$, the "thermal wind") in the direction of the temperature gradient along the neutral tangent plane $\nabla_n \Theta$, since, from Eqn. (3.12.3), namely $f\,\mathbf{v}_z = \frac{N^2}{g\rho}\,\mathbf{k}\times\nabla_n P$, and the third line of (3.13.2), namely $H^{\mathrm{n}} = g^{-1} N^2 T_{\mathrm{b}}^{\Theta}\left(\nabla_n P \times \nabla_n \Theta\right)\cdot\mathbf{k}$, we find that

$$H^{\mathrm{n}} = \rho T_{\mathrm{b}}^{\Theta} f \mathbf{v}_z \cdot \nabla_n \Theta. \tag{3.13.4}$$

This connection between neutral helicity and an aspect of the horizontal velocity vector motivates the idea that the mean velocity might be somehow linked to neutral helicity, and this link is established in this section.

The absolute velocity vector in the ocean can be written as a closed expression involving the neutral helicity, and this expression is derived as follows. First the water-mass transformation equation (A.23.1) is written as

$$\begin{aligned}\bar{\mathbf{v}}\cdot\nabla_n\hat{\Theta} \;&=\; \gamma_z \nabla_n\cdot\left(\gamma_z^{-1} K \nabla_n \hat{\Theta}\right) + KgN^{-2}\hat{\Theta}_z\left(C_{\mathrm{b}}^{\Theta}\nabla_n\hat{\Theta}\cdot\nabla_n\hat{\Theta} + T_{\mathrm{b}}^{\Theta}\nabla_n\hat{\Theta}\cdot\nabla_n P\right)\\ &\quad + D\beta^{\Theta} g N^{-2}\hat{\Theta}_z^3 \frac{d^2\hat{S}_{\mathrm{A}}}{d\hat{\Theta}^2} \;-\; \mathbf{\Psi}_z\cdot\nabla_n\hat{\Theta} \;-\; \hat{\Theta}_t\big|_n ,\end{aligned} \tag{3.13.7}$$

where the thickness-weighted mean velocity of density-coordinate averaging, $\hat{\mathbf{v}}$, has been written as $\hat{\mathbf{v}} = \bar{\mathbf{v}} + \mathbf{\Psi}_z$, that is, as the sum of the Eulerian-mean horizontal velocity $\bar{\mathbf{v}}$ and the quasi-Stokes eddy-induced horizontal velocity $\mathbf{\Psi}_z$ (McDougall and McIntosh (2001)). The quasi-Stokes vector streamfunction $\mathbf{\Psi}$ is usually expressed in terms of an imposed lateral diffusivity and the slope of the locally-referenced potential density surface (Gent *et al.*, (1995)). More generally, at least in a steady state when $\hat{\Theta}_t\big|_n$ is zero, the right-hand side of Eqn. (3.13.7) is due only to mixing processes and once the form of the lateral and vertical diffusivities are known, these terms are known in terms of the ocean's hydrography. Eqn. (3.13.9) is written more compactly as

$$\bar{\mathbf{v}}\cdot\boldsymbol{\tau} \;=\; v^{\perp} \qquad \text{where} \qquad \boldsymbol{\tau} \;\equiv\; \nabla_n\hat{\Theta}\big/\left|\nabla_n\hat{\Theta}\right| , \tag{3.13.8}$$

and $v^{\perp}$ is interpreted as being due to mixing processes.

The mean horizontal velocity $\bar{\mathbf{v}}$ is now split into the components along and across the contours of $\hat{\Theta}$ on the neutral tangent plane so that

$$\bar{\mathbf{v}} \;=\; v^{\|}\,\boldsymbol{\tau}\times\mathbf{k} \;+\; v^{\perp}\boldsymbol{\tau} \;, \tag{3.13.9}$$

where $v^{\|} = \bar{\mathbf{v}}\cdot\boldsymbol{\tau}\times\mathbf{k}$. Note that if $\boldsymbol{\tau}$ points northwards then $\boldsymbol{\tau}\times\mathbf{k}$ points eastward. The expression $\bar{\mathbf{v}}\cdot\boldsymbol{\tau} = v^{\perp}$ of Eqn. (3.13.8) is now vertically differentiated to obtain

$$\bar{\mathbf{v}}\cdot\boldsymbol{\tau}_z \;=\; -\bar{\mathbf{v}}_z\cdot\boldsymbol{\tau} \;+\; v_z^{\perp} \;=\; -\tfrac{N^2}{fg\rho}\,\mathbf{k}\times\nabla_n P\cdot\boldsymbol{\tau} \;+\; v_z^{\perp} , \tag{3.13.10}$$

where we have used the "thermal wind" equation (3.12.3), $\bar{\mathbf{v}}_z = \frac{N^2}{fg\rho}\,\mathbf{k}\times\nabla_n P$. We will now show that the left-hand side of this equation is $-\phi_z v^{\|}$ where ϕ_z is the rate of rotation of the direction of the unit vector $\boldsymbol{\tau}$ with respect to height (in radians per metre). By expressing the two-dimensional unit vector $\boldsymbol{\tau}$ in terms of the angle ϕ (measured counter-clockwise) of $\boldsymbol{\tau}$ with respect to due north so that $\boldsymbol{\tau} = \left(-\sin\phi, \cos\phi\right)$, we see that $\boldsymbol{\tau}\times\mathbf{k} = \left(\cos\phi, \sin\phi\right)$, $\boldsymbol{\tau}_z = -\phi_z\,\boldsymbol{\tau}\times\mathbf{k}$ and $\mathbf{k}\cdot\boldsymbol{\tau}\times\boldsymbol{\tau}_z = \phi_z$. Interestingly, ϕ_z is also equal to minus the helicity of $\boldsymbol{\tau}$ (and to minus the helicity of $\boldsymbol{\tau}\times\mathbf{k}$), that is, $\phi_z = -\,\boldsymbol{\tau}\cdot\nabla\times\boldsymbol{\tau} = -\left(\boldsymbol{\tau}\times\mathbf{k}\right)\cdot\nabla\times\left(\boldsymbol{\tau}\times\mathbf{k}\right)$, where the helicity of a vector is defined to be the scalar product of the vector with its curl. From the velocity decomposition (3.13.9) and the equation $\boldsymbol{\tau}_z = -\phi_z\,\boldsymbol{\tau}\times\mathbf{k}$ we see that the left-hand side of Eqn. (3.13.10), $\bar{\mathbf{v}}\cdot\boldsymbol{\tau}_z$, is $-\phi_z v^{\|}$, hence $v^{\|}$ can be expressed as

$$v^{\|} \;=\; \frac{N^2}{fg\rho}\,\frac{\mathbf{k}\cdot\nabla_n P\times\boldsymbol{\tau}}{\phi_z} \;-\; \frac{v_z^{\perp}}{\phi_z} \qquad \text{or} \qquad v^{\|} \;=\; \frac{H^{\mathrm{n}}}{\phi_z\,\rho f\, T_{\mathrm{b}}^{\Theta}\left|\nabla_n\hat{\Theta}\right|} \;-\; \frac{v_z^{\perp}}{\phi_z} , \tag{3.13.11}$$

where we have used the definition of neutral helicity H^{n}, Eqn. (3.13.2). The full expression for both horizontal components of the mean horizontal velocity vector $\overline{\mathbf{v}}$ is then

$$\boxed{\overline{\mathbf{v}} \;=\; \left\{ \frac{N^2}{fg\rho} \, \frac{\mathbf{k}\cdot\nabla_n P\times\boldsymbol{\tau}}{\phi_z} \;-\; \frac{v_z^{\perp}}{\phi_z} \right\} \frac{\nabla_n\hat{\Theta}}{\left|\nabla_n\hat{\Theta}\right|}\times\mathbf{k} \;+\; v^{\perp}\frac{\nabla_n\hat{\Theta}}{\left|\nabla_n\hat{\Theta}\right|} .} \qquad (3.13.12)$$

Neutral helicity arises in this context because it is proportional to the component of the thermal wind vector $\overline{\mathbf{v}}_z$ in the direction across the $\hat{\Theta}$ contour on the neutral tangent plane (see Eqn. (3.13.4)).

This equation (3.13.12) for the Eulerian-mean horizontal velocity $\overline{\mathbf{v}}$ shows that in the absence of mixing processes (so that $v^{\perp} = v_z^{\perp} = 0$) and so long as (i) the epineutral $\hat{\Theta}$ contours do spiral in the vertical and (ii) $\left|\nabla_n\hat{\Theta}\right|$ is not zero, then neutral helicity H^{n} (which is proportional to $\mathbf{k}\cdot\nabla_n P\times\boldsymbol{\tau}$) is required to be non-zero in the ocean whenever the ocean is not motionless.

Planetary potential vorticity

Planetary potential vorticity is the Coriolis parameter f times the vertical gradient of a suitable variable. Potential density is sometimes used for that variable but using potential density (i) involves an inaccurate separation between lateral and diapycnal advection because potential density surfaces are not a good approximation to neutral tangent planes and (ii) incurs the non-conservative baroclinic production term of Eqn. (3.13.4). Using approximately neutral surfaces, "ans", (such as Neutral Density surfaces) provides an optimal separation between the effects of lateral and diapycnal mixing in the potential vorticity equation. In this case the potential vorticity variable is proportional to the reciprocal of the thickness between a pair of closely spaced approximately neutral surfaces.

The evolution equation for planetary potential vorticity is derived by first taking the epineutral "divergence" $\nabla_n\cdot$ of the geostrophic relationship from Eqn. (3.12.1), namely $f\mathbf{v} \;=\; g\,\mathbf{k}\times\nabla_p z$. The projected "divergences" of a two-dimensional vector $\mathbf{a}$ in the neutral tangent plane and in an isobaric surface, are $\nabla_n\cdot\mathbf{a} \;=\; \nabla_z\cdot\mathbf{a} + \mathbf{a}_z\cdot\nabla_n z$ and $\nabla_p\cdot\mathbf{a} \;=\; \nabla_z\cdot\mathbf{a} + \mathbf{a}_z\cdot\nabla_p z$ from which we find (using Eqn. (3.12.6), $\nabla_n z - \nabla_p z \;=\; \nabla_n P/P_z$)

$$\nabla_n\cdot\mathbf{a} \;=\; \nabla_p\cdot\mathbf{a} + \mathbf{a}_z\cdot\nabla_n P/P_z . \qquad (3.20.1)$$

Applying this relationship to the two-dimensional vector $f\mathbf{v} \;=\; g\,\mathbf{k}\times\nabla_p z$ we have

$$\nabla_n\cdot\left(f\mathbf{v}\right) \;=\; g\,\nabla_p\cdot\left(\mathbf{k}\times\nabla_p z\right) \;+\; f\mathbf{v}_z\cdot\nabla_n P/P_z \;=\; 0 . \qquad (3.20.2)$$

The first part of this expression can be seen to be zero by simply calculating its components, and the second part is zero because the thermal wind vector $\mathbf{v}_z$ points in the direction $\mathbf{k}\times\nabla_n P$ (see Eqn. (3.12.3)). It can be shown that $\nabla_r\cdot\left(f\mathbf{v}\right) \;=\; 0$ in any surface r which contains the line $\nabla P\times\nabla\rho$.

Eqn. (3.20.2), namely $\nabla_n\cdot\left(f\mathbf{v}\right) \;=\; 0$, can be interpreted as the divergence form of the evolution equation of planetary potential vorticity since

$$\nabla_n\cdot\left(f\mathbf{v}\right) \;=\; \nabla_n\cdot\left(\frac{q\,\mathbf{v}}{\gamma_z}\right) \;=\; 0 \; , \qquad (3.20.3)$$

where $q = f\gamma_z$ is the planetary potential vorticity, being the Coriolis parameter times the vertical gradient of Neutral Density. This instantaneous equation can be averaged in a thickness-weighted sense in density coordinates yielding

$$\boxed{\nabla_n\cdot\left(\frac{\hat{q}\,\hat{\mathbf{v}}}{\tilde{\gamma}_z}\right) \;=\; -\,\nabla_n\cdot\overline{\left(\frac{\mathbf{v}''q''}{\gamma_z}\right)} \;=\; \nabla_n\cdot\left(\tilde{\gamma}_z^{-1}K\nabla_n\hat{q}\right)} , \qquad (3.20.4)$$

where the double-primed quantities are deviations of the instantaneous values from the thickness-weighted mean quantities. Here the epineutral eddy flux of planetary potential vorticity per unit area has been taken to be down the epineutral gradient of $\hat{q}$ with the epineutral diffusivity K. The thickness-weighted mean planetary potential vorticity is

$$\hat{q} \;\equiv\; \tilde{\gamma}_z \overline{\left(\frac{q}{\gamma_z}\right)\Bigg|_{\gamma}} \;=\; f\tilde{\gamma}_z \;, \qquad (3.20.5)$$

and the averaging in the above equations is consistent with the difference between the thickness-weighted mean velocity and the velocity averaged *on* the Neutral Density surface, $\hat{\mathbf{v}} - \tilde{\mathbf{v}}$ (the bolus velocity), being $\hat{\mathbf{v}} - \tilde{\mathbf{v}} \;=\; K\nabla_n \ln\left(\hat{q}\right)$, since Eqn. (3.20.4) can be written as $\nabla_n \cdot \left(f\hat{\mathbf{v}}\right) = \nabla_n \cdot \left(\tilde{\gamma}_z^{-1} K \nabla_n \hat{q}\right)$ while the average of Eqn. (3.20.3) is $\nabla_n \cdot \left(f\tilde{\mathbf{v}}\right) = 0$.

The divergence form of the mean planetary potential vorticity evolution equation, Eqn. (3.20.4), is quite different to that of a normal conservative variable such as Absolute Salinity or Conservative Temperature,

$$\left(\frac{\hat{\Theta}}{\tilde{\gamma}_z}\bigg|_n\right)_t \;+\; \nabla_n \cdot \left(\frac{\hat{\Theta}\hat{\mathbf{v}}}{\tilde{\gamma}_z}\right) \;+\; \frac{\left(\tilde{e}\hat{\Theta}\right)_z}{\tilde{\gamma}_z} \;=\; \nabla_n \cdot \left(\tilde{\gamma}_z^{-1} K \nabla_n \hat{\Theta}\right) \;+\; \frac{\left(D\hat{\Theta}_z\right)_z}{\tilde{\gamma}_z} \;, \qquad (\hat{\Theta}\text{\_Eqn.})$$

because in Eqn. (3.20.4) the following three terms are missing; (i) the vertical diffusion of $\hat{q}$ with diffusivity D (ii) the dianeutral advection of $\hat{q}$ by the dianeutral velocity $\tilde{e}$, and (iii) the temporal tendency term.

The mean planetary potential vorticity equation (3.20.4) may be put into the advective form by subtracting $\hat{q}$ times the mean continuity equation,

$$\left(\frac{1}{\tilde{\gamma}_z}\bigg|_n\right)_t \;+\; \nabla_n \cdot \left(\frac{\hat{\mathbf{v}}}{\tilde{\gamma}_z}\right) \;+\; \frac{\tilde{e}_z}{\tilde{\gamma}_z} \;=\; 0 \;, \qquad (3.20.6)$$

from Eqn. (3.20.4), yielding

$$\hat{q}_t\big|_n \;+\; \hat{\mathbf{v}} \cdot \nabla_n \hat{q} \;=\; \tilde{\gamma}_z \nabla_n \cdot \left(\tilde{\gamma}_z^{-1} K \nabla_n \hat{q}\right) \;+\; \hat{q}\tilde{e}_z \;, \qquad (3.20.7)$$

or

$$\boxed{\hat{q}_t\big|_n \;+\; \hat{\mathbf{v}} \cdot \nabla_n \hat{q} \;+\; \tilde{e}\hat{q}_z \;=\; \frac{\mathrm{d}\hat{q}}{\mathrm{d}t} \;=\; \tilde{\gamma}_z \nabla_n \cdot \left(\tilde{\gamma}_z^{-1} K \nabla_n \hat{q}\right) \;+\; \left(\hat{q}\tilde{e}\right)_z}\,. \qquad (3.20.8)$$

In this form, it is clear that potential vorticity behaves like a conservative variable as far as epineutral mixing is concerned, but it is quite unlike a normal conservative variable as far as vertical mixing is concerned; contrast Eqn. (3.20.8) with the conservation equation for Conservative Temperature,

$$\boxed{\hat{\Theta}_t\big|_n \;+\; \hat{\mathbf{v}} \cdot \nabla_n \hat{\Theta} \;+\; \tilde{e}\hat{\Theta}_z \;=\; \frac{\mathrm{d}\hat{\Theta}}{\mathrm{d}t} \;=\; \gamma_z \nabla_n \cdot \left(\gamma_z^{-1} K \nabla_n \hat{\Theta}\right) + \left(D\hat{\Theta}_z\right)_z}\,. \qquad \underline{(\text{A.21.15})}$$

If $\hat{q}$ were a normal conservative variable the last term in Eqn. (3.20.8) would be $\left(D\hat{q}_z\right)_z$ where D is the vertical diffusivity. The term that actually appears in Eqn. (3.20.8), $\left(\hat{q}\tilde{e}\right)_z$, is different to $\left(D\hat{q}_z\right)_z$ by $\left(\hat{q}\tilde{e} - D\hat{q}_z\right)_z \;=\; f\left(\tilde{e}\tilde{\gamma}_z - D\tilde{\gamma}_{zz}\right)_z$. Equation (A.22.4) for the mean dianeutral velocity $\tilde{e}$ can be expressed as $\tilde{e} \approx D_z \;+\; D\tilde{\gamma}_{zz}/\tilde{\gamma}_z$ if the following three aspects of the non-linear equation of state are ignored; (1) cabbeling and thermobaricity, (2) the vertical variation of the thermal expansion coefficient and the saline contraction coefficient, and (3) the vertical variation of the integrating factor $b\left(x,y,z\right)$ of Eqns. (3.20.10) - (3.20.15) below. Even when ignoring these three different implications of the nonlinear equation of state, the evolution equations (3.20.7) and (3.20.8) of $\hat{q}$ are unlike normal conservation equations because of the extra term

$$\left(\hat{q}\tilde{e} - D\hat{q}_z\right)_z \;=\; f\left(\tilde{e}\tilde{\gamma}_z - D\tilde{\gamma}_{zz}\right)_z \;\approx\; f\left(D_z\tilde{\gamma}_z\right)_z \;=\; \left(D_z\hat{q}\right)_z \qquad (3.20.9)$$

on their right-hand sides. This presence of this additional term can result in "unmixing" of $\hat{q}$ in the vertical. Consider a situation where both $\hat{q}$ and $\hat{\Theta}$ are locally linear functions of $\hat{S}_{\mathrm{A}}$ down a vertical water column, so that the $\hat{S}_{\mathrm{A}} - \hat{q}$ and $\hat{S}_{\mathrm{A}} - \hat{\Theta}$ diagrams are both locally straight lines, exhibiting no curvature. Imposing a large amount of vertical mixing at one height (e. g. a delta function of D) will not change the $\hat{S}_{\mathrm{A}} - \hat{\Theta}$ diagram because of the zero $\hat{S}_{\mathrm{A}} - \hat{\Theta}$ curvature (see the water-mass transformation equation (A.23.1)). However, the additional term $\left(D_z \hat{q}\right)_z$ of Eqn. (3.20.9) means that there will be a change in $\hat{q}$ of $\left(D_z \hat{q}\right)_z = \hat{q} D_{zz} + \hat{q}_z D_z \approx \hat{q} D_{zz}$. This is $\hat{q}$ times a negative anomaly at the central height of the extra vertical diffusion, and is $\hat{q}$ times a positive anomaly on the flanking heights above and below the central height. In this way, a delta function of extra vertical diffusion induces structure in the initially straight $\hat{S}_{\mathrm{A}} - \hat{q}$ line which is a telltale sign of "unmixing".

This planetary potential vorticity variable, $\hat{q} = f \tilde{\gamma}_z$, is often mapped on Neutral Density surfaces to give insight into the mean circulation of the ocean on density surfaces. The reasoning is that if the influence of dianeutral advection (the last term in Eqn. (3.20.7)) is small, and the epineutral mixing of $\hat{q}$ is also small, then in a steady ocean $\hat{\mathbf{v}} \cdot \nabla_n \hat{q} = 0$ and the thickness-weighted mean flow on density surfaces $\hat{\mathbf{v}}$ will be along contours of thickness-weighted planetary potential vorticity $\hat{q} = f \tilde{\gamma}_z$.

Because the square of the buoyancy frequency, N^2, accurately represents the vertical static stability of a water column, there is a strong urge to regard fN^2 as the appropriate planetary potential vorticity variable, and to map its contours on Neutral Density surfaces. This urge must be resisted, as spatial maps of fN^2 are significantly different to those of $\hat{q} = f \tilde{\gamma}_z$. To see why this is the case the relationship between the epineutral gradients of $\hat{q}$ and fN^2 will be derived.

For the present purposes Neutral Helicity will be assumed sufficiently small that the existence of neutral surfaces is a good approximation, and we seek the integrating factor $b = b(x,y,z)$ which allows the construction of Neutral Density surfaces (γ surfaces) according to

$$\frac{\nabla \gamma}{\gamma} = b\left(\beta^{\Theta} \nabla S_{\mathrm{A}} - \alpha^{\Theta} \nabla \Theta\right) = b\left(\frac{\nabla \rho}{\rho} - \kappa \nabla P\right). \tag{3.20.10}$$

Taking the curl of this equation gives

$$\frac{\nabla b}{b} \times \left(\kappa \nabla P - \frac{\nabla \rho}{\rho}\right) = -\nabla \kappa \times \nabla P. \tag{3.20.11}$$

The bracket on the left-hand side is normal to the neutral tangent plane and points in the direction $\mathbf{n} = -\nabla_n z + \mathbf{k}$ and is $g^{-1} N^2 \left(-\nabla_n z + \mathbf{k}\right)$. Taking the component of Eqn. (3.20.11) in the direction of the normal to the neutral tangent plane, $\mathbf{n}$, we find

$$\begin{aligned} 0 &= \nabla \kappa \times \nabla P \cdot \mathbf{n} = \left(\nabla_n \kappa + \kappa_z \mathbf{n}\right) \times \left(\nabla_n P + P_z \mathbf{n}\right) \cdot \mathbf{n} \\ &= \nabla_n \kappa \times \nabla_n P \cdot \mathbf{n} = \nabla_n \kappa \times \nabla_n P \cdot \mathbf{k} = \left(\kappa_{S_{\mathrm{A}}} \nabla_n S_{\mathrm{A}} + \kappa_{\Theta} \nabla_n \Theta\right) \times \nabla_n P \cdot \mathbf{k} \\ &= T_{\mathrm{b}}^{\Theta} \nabla_n P \times \nabla_n \Theta \cdot \mathbf{k} = g N^{-2} H^{\mathrm{n}}, \end{aligned} \tag{3.20.12}$$

which simply says that the neutral helicity H^{n} must be zero in order for the dianeutral component of Eqn. (3.20.11) to hold, that is, $\nabla_n P \times \nabla_n \Theta \cdot \mathbf{k}$ must be zero. Here the equalities $\kappa_{S_{\mathrm{A}}} = \beta_P^{\Theta}$ and $\kappa_{\Theta} = -\alpha_P^{\Theta}$ have been used.

Since ∇b can be written as $\nabla b = \nabla_n b + b_z \mathbf{n}$, Eqn. (3.20.11) becomes

$$g^{-1} N^2 \, \nabla_n \ln b \times \left(-\nabla_n z + \mathbf{k}\right) = -P_z \nabla_p \kappa \times \left(-\nabla_p z + \mathbf{k}\right), \tag{3.20.13}$$

where $\nabla P = P_z \left(-\nabla_p z + \mathbf{k}\right)$ has been used on the right-hand side, $\left(-\nabla_p z + \mathbf{k}\right)$ being the normal to the isobaric surface. Concentrating on the horizontal components of this equation, $g^{-1}N^2\,\nabla_n \ln b \;=\; -\,P_z\,\nabla_p\kappa$, and using the hydrostatic equation $P_z = -\,g\rho$ gives

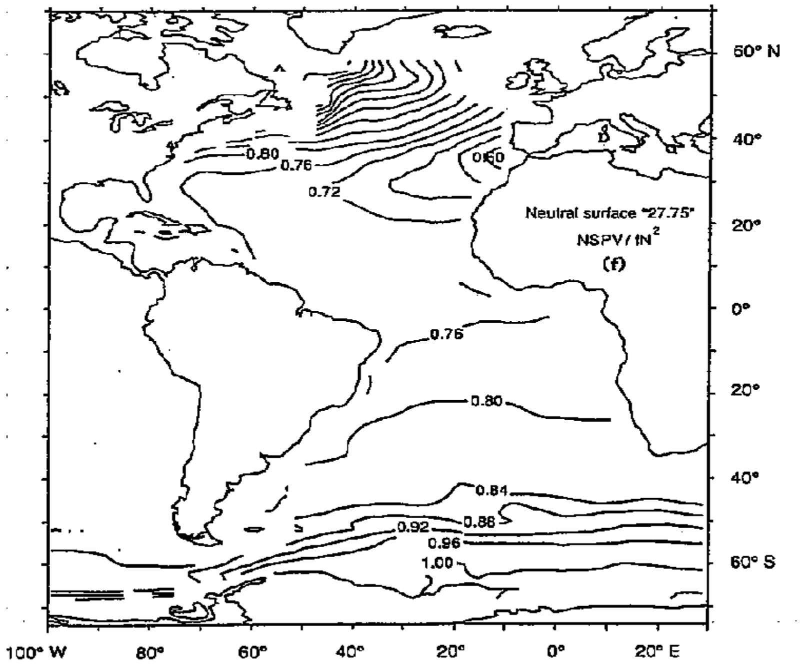
$$\boxed{\nabla_n \ln b \;=\; \rho g^2 N^{-2}\,\nabla_p\kappa \;=\; -\,\rho g^2 N^{-2}\left(\alpha_P^\Theta \nabla_p \Theta - \beta_P^\Theta \nabla_p S_\mathrm{A}\right)}. \qquad (3.20.14)$$

The integrating factor b defined by Eqn. (3.20.10), that is $b \equiv \left(\rho^l/\gamma\right)\nabla\gamma\cdot\nabla\rho^l\big/(\nabla\rho^l\cdot\nabla\rho^l)$ where $\nabla\rho^l \equiv \rho^l(\beta^\Theta\nabla S_\mathrm{A} - \alpha^\Theta\nabla\Theta)$, allows spatial integrals of $b\,(\beta^\Theta\nabla S_\mathrm{A} - \alpha^\Theta\nabla\Theta) \;=\; b\,\nabla\ln\rho^l \;\approx\; \nabla\ln\gamma$ to be approximately independent of path for "vertical paths", that is, for paths in surfaces whose normal has zero vertical component.

By analogy with fN^2, the Neutral Surface Potential Vorticity ($NSPV$) is defined as $-g\gamma^{-1}$ times $\hat{q} \;=\; f\tilde{\gamma}_z$, so that $NSPV = b\,fN^2$ (having used the vertical component of Eqn. (3.20.9)), so that the ratio of $NSPV$ to fN^2 is found from Eqn. (3.20.14) to be

$$\begin{aligned}\frac{NSPV}{fN^2} \;=\; b \;=\; \frac{\rho^l}{\gamma}\frac{\gamma_z}{\rho_z^l} \;&=\; \exp\left\{-\int_\mathrm{ans}\rho g^2\,N^{-2}\left(\alpha_P^\Theta\nabla_p\Theta - \beta_P^\Theta\nabla_p S_\mathrm{A}\right)\cdot d\boldsymbol{l}\right\} \\ &=\; \exp\left\{\int_\mathrm{ans}\rho g^2\,N^{-2}\nabla_p\kappa\cdot d\boldsymbol{l}\right\}.\end{aligned} \qquad (3.20.15)$$

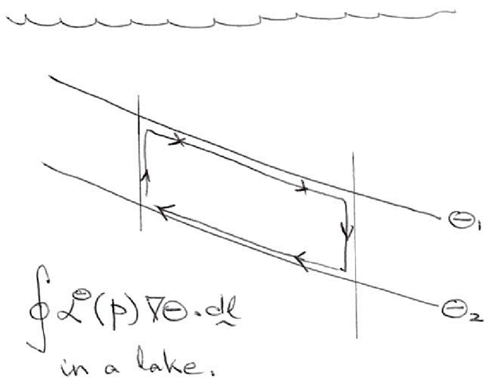
The integral here is taken along an approximately neutral surface (such a Neutral Density surface) from a location where $NSPV$ is equal to fN^2.



The deficiencies of fN^2 as a form of planetary potential vorticity have not been widely appreciated. Even in a lake, the use of fN^2 as planetary potential vorticity is inaccurate since the right-hand side of (3.20.14) is then

$$-\rho g^2 N^{-2}\alpha_P^\Theta\nabla_p\Theta \;=\; \rho g^2 N^{-2}\alpha_P^\Theta\Theta_z\,\nabla_\Theta P/P_z \;=\; -\,\frac{\alpha_P^\Theta}{\alpha^\Theta}\,\nabla_\Theta P, \qquad (3.20.16)$$

where the geometrical relationship $\nabla_p\Theta = -\,\Theta_z\,\nabla_\Theta P/P_z$ has been used along with the hydrostatic equation. The mere fact that the Conservative Temperature surfaces in the lake have a slope (i. e. $\nabla_\Theta P \neq \mathbf{0}$) means that the spatial variation of contours of fN^2 will not be the same as for the contours of $NSPV$ in a lake.



In the situation where there is no gradient of Conservative Temperature along a Neutral Density surface ($\nabla_\gamma \Theta = \mathbf{0}$) the contours of $NSPV$ along the Neutral Density surface coincide with those of isopycnal-potential-vorticity (IPV), the potential vorticity defined with respect to the vertical gradient of potential density by $IPV = -fg\rho^{-1}\rho_z^{\Theta}$. IPV is related to fN^2 by (McDougall (1988))

$$\frac{IPV}{fN^2} \equiv \frac{-g\rho^{-1}\rho_z^{\Theta}}{N^2} = \frac{\beta^{\Theta}(p_\mathrm{r})}{\beta^{\Theta}(p)}\frac{\left[R_\rho/r-1\right]}{\left[R_\rho-1\right]} = \frac{\beta^{\Theta}(p_\mathrm{r})}{\beta^{\Theta}(p)}\frac{1}{G^{\Theta}} \approx \frac{1}{G^{\Theta}}, \qquad (3.20.17)$$

so that the ratio of $NSPV$ to IPV plotted on an approximately neutral surface is given by

$$\frac{NSPV}{IPV} = \frac{\beta^{\Theta}(p)}{\beta^{\Theta}(p_\mathrm{r})}\frac{\left[R_\rho-1\right]}{\left[R_\rho/r-1\right]}\exp\left\{\int_{\mathrm{ans}} g^2 N^{-2}\nabla_P(\rho\kappa)\cdot d\boldsymbol{l}\right\}. \qquad (3.20.18)$$

The sketch below indicates why $NSPV$ is different to IPV; it is because of the highly differentiated nature of potential vorticity that isolines of IPV and $NSPV$ do not coincide even at the reference pressure p_r of the potential density variable. $NSPV$, fN^2 and IPV have the units s^{-3}.

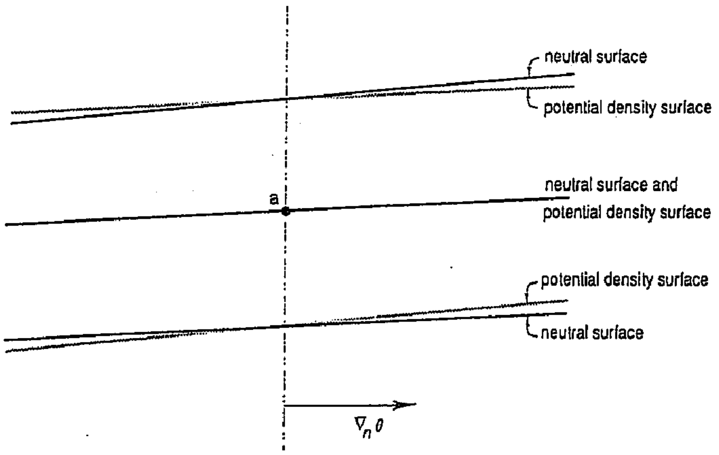


Fig. 14. A vertical cross section through three neutral surfaces and three potential density surfaces. The reference pressure of the potential density is the pressure of the central point, a. The neutral surface and potential density surface that pass though this point are parallel. The slopes of the other pairs of surfaces are different.