

Double-diffusive convection; “salt-fingers”

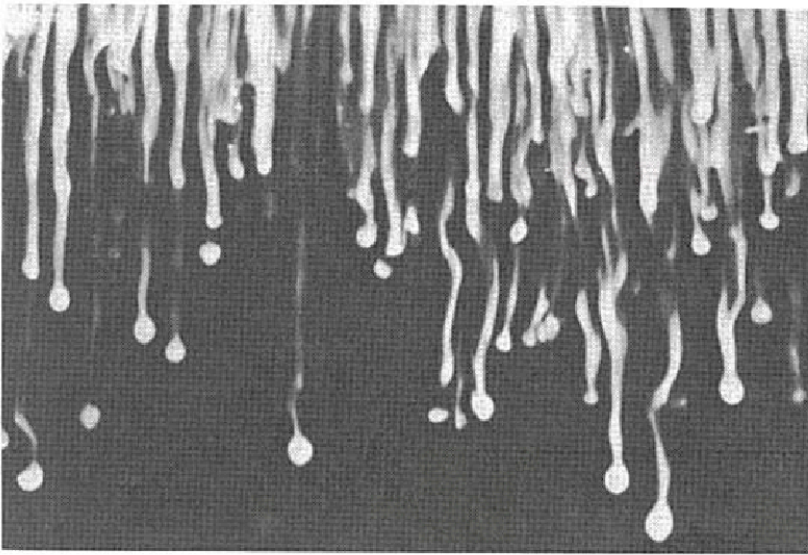


Figure 4.17. Salt fingers. These are formed in the laboratory by carefully allowing a warm weak brine solution containing a fluorescent dye to spread over a cold fresh layer of water. A vertical sheet of light is used to cause the dye to emit light and so make the developing fingers visible in this horizontal view. The ‘fingers’ are typically 2 mm in width in the laboratory experiment, but in the ocean fingers may be some few centimetres wide. (From Huppert and Turner, 1981.)

The “budget method” of estimating the vertical diffusivity D

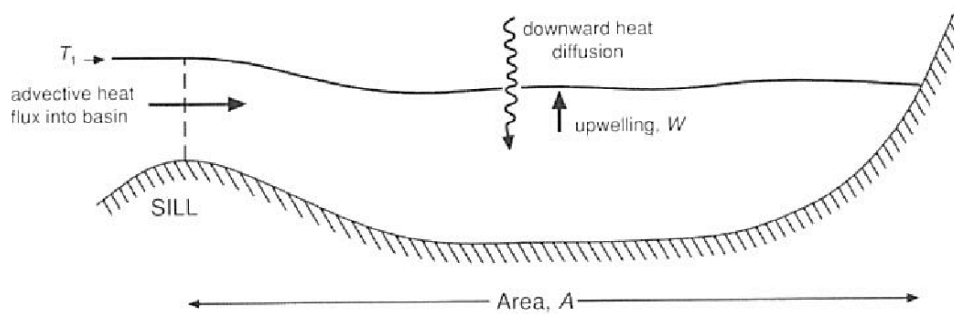


Figure 4.22. Heat balance in a deep-ocean basin. Flow enters a deep-ocean basin over a sill on the left. The basin is closed on the right. There is a downward flux of heat by diffusion from water of higher temperature above the T_1 isotherm and upward (upwelling) motion at speed, w , within the basin at the level of the T_1 isotherm, maintaining a steady state.

This “budget method” is a way of estimating the vertical diffusivity across the isotherm without measuring the properties of the turbulence at the centimeter scale.

The water-mass transformation equation

It is instructive to substitute Eqn. (A.22.4) for \tilde{e} into the expression (A.21.15) for the material derivative of $\hat{\Theta}$, thus eliminating \tilde{e} and obtaining the following equation for the temporal and spatial evolution of $\hat{\Theta}$ along the neutral tangent plane (McDougall (1984))

$$\left. \hat{\Theta}_t \right|_n + \hat{\mathbf{v}} \cdot \nabla_n \hat{\Theta} = \gamma_z \nabla_n \cdot (\gamma_z^{-1} K \nabla_n \hat{\Theta}) + K g N^{-2} \hat{\Theta}_z \left(C_b^\Theta \nabla_n \hat{\Theta} \cdot \nabla_n \hat{\Theta} + T_b^\Theta \nabla_n \hat{\Theta} \cdot \nabla_n P \right) + D \beta^\Theta g N^{-2} \hat{\Theta}_z^3 \frac{d^2 \hat{S}_A}{d\hat{\Theta}^2}. \quad (\text{A.23.1})$$

The term involving D has been written as proportional to the curvature of the $\hat{S}_A - \hat{\Theta}$ diagram of a vertical cast; this term can also be written as $D \beta^\Theta g N^{-2} (\hat{\Theta}_z \hat{S}_{A_{zz}} - \hat{S}_{A_z} \hat{\Theta}_{zz})$. The form of Eqn. (A.23.1) illustrates that when analyzed in density coordinates, Conservative Temperature (and Absolute Salinity) (i) are affected not only by the expected lateral diffusion process along density surfaces but also by the nonlinear dianeutral advection processes, cabbeling and thermobaricity, (ii) are affected by diapycnal turbulent mixing only to the extent that the vertical $\hat{S}_A - \hat{\Theta}$ diagram is not locally straight, and (iii) are not influenced by the vertical variation of D since D_z does not appear in this equation.

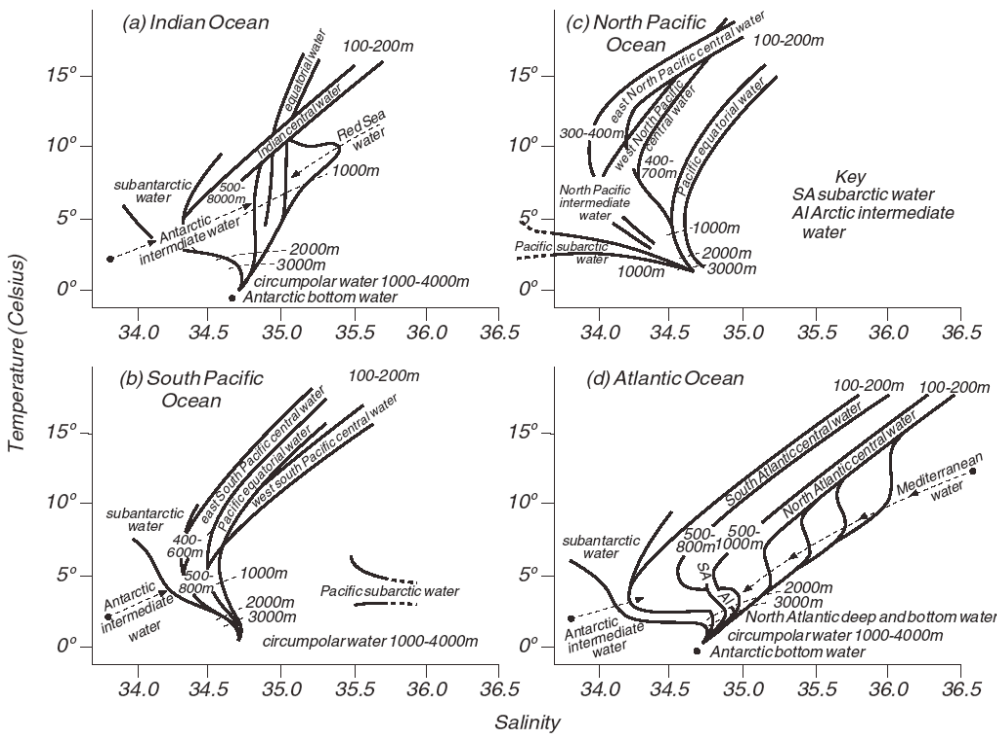


Figure 13.11 T - S plots of water in the various ocean basins. After Tolmazin (1985: 138).

A review of our basic conservation equations

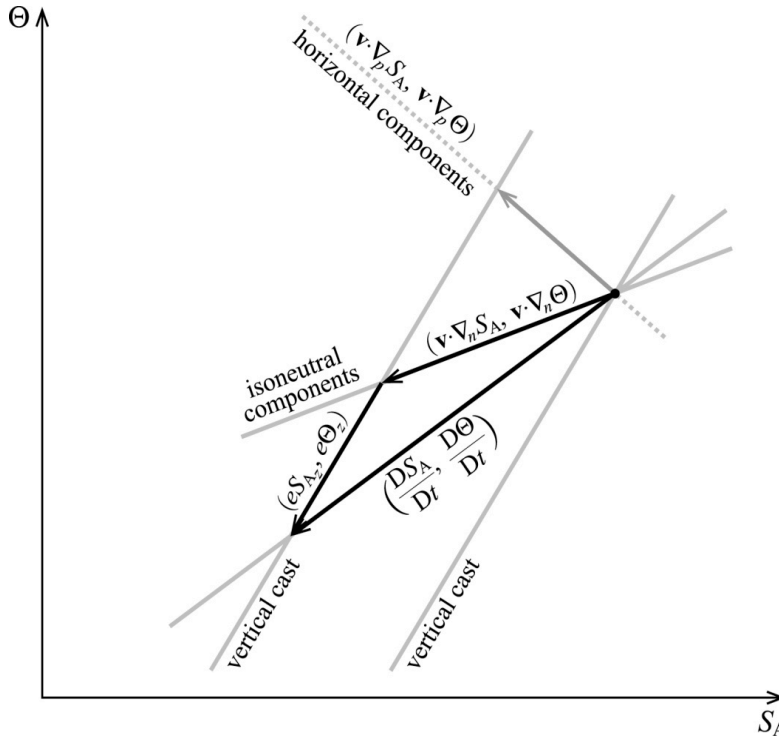
$$\left. \frac{\partial \hat{S}_A}{\partial t} \right|_n + \hat{\mathbf{v}} \cdot \nabla_n \hat{S}_A + \tilde{e} \frac{\partial \hat{S}_A}{\partial z} = \gamma_z \nabla_n \cdot \left(\gamma_z^{-1} K \nabla_n \hat{S}_A \right) + \left(D \frac{\partial \hat{S}_A}{\partial z} \right)_z. \quad (\text{A.21.11})$$

$$\left. \hat{\Theta} \right|_n + \hat{\mathbf{v}} \cdot \nabla_n \hat{\Theta} + \tilde{e} \frac{\partial \hat{\Theta}}{\partial z} = \gamma_z \nabla_n \cdot \left(\gamma_z^{-1} K \nabla_n \hat{\Theta} \right) + \left(D \hat{\Theta}_z \right)_z. \quad (\text{A.21.15})$$

$$\tilde{e} g^{-1} N^2 = -K \left(C_b^\Theta \nabla_n \hat{\Theta} \cdot \nabla_n \hat{\Theta} + T_b^\Theta \nabla_n \hat{\Theta} \cdot \nabla_n P \right) + \alpha^\Theta \left(D \hat{\Theta}_z \right)_z - \beta^\Theta \left(D \hat{S}_{A_z} \right)_z. \quad (\text{A.22.4})$$

$$\left. \hat{\Theta} \right|_n + \hat{\mathbf{v}} \cdot \nabla_n \hat{\Theta} = \gamma_z \nabla_n \cdot \left(\gamma_z^{-1} K \nabla_n \hat{\Theta} \right) + K g N^{-2} \hat{\Theta}_z \left(C_b^\Theta \nabla_n \hat{\Theta} \cdot \nabla_n \hat{\Theta} + T_b^\Theta \nabla_n \hat{\Theta} \cdot \nabla_n P \right) + D \beta^\Theta g N^{-2} \hat{\Theta}_z^3 \frac{d^2 \hat{S}_A}{d \hat{\Theta}^2}. \quad (\text{A.23.1})$$

Equations (A.21.11) and (A.21.15) are the fundamental evolution equations of Absolute Salinity and Conservative Temperature in a turbulent ocean, and the pair of equations (A.22.4) and (A.23.1) are simply derived as linear combinations of Eqns. (A.21.11) and (A.21.15). The “density” conservation equation (A.22.4) and the “water-mass transformation” equation (A.23.1) are in some sense the “normal modes” of Eqns. (A.21.11) and (A.21.15). That is, Eqn. (A.22.4) expresses how mixing processes contribute to the mean vertical velocity \tilde{e} through the neutral tangent plane, while (A.23.1) expresses how the tracer called “Conservative Temperature measured along the neutral direction” is affected by mixing processes; this equation does not contain \tilde{e} .



For completeness, the water-mass conservation equation for Absolute Salinity that corresponds to Eqn. (A.23.1) is

$$\left. \frac{\partial \hat{S}_A}{\partial t} \right|_n + \hat{\mathbf{v}} \cdot \nabla_n \hat{S}_A = \gamma_z \nabla_n \cdot (\gamma_z^{-1} K \nabla_n \hat{S}_A) + K g N^{-2} \hat{S}_{A_z} (C_b^\Theta \nabla_n \hat{\Theta} \cdot \nabla_n \hat{\Theta} + T_b^\Theta \nabla_n \hat{\Theta} \cdot \nabla_n P) \quad (\text{A.23.2})$$

$$+ D \alpha^\Theta g N^{-2} \hat{\Theta}_z^3 \frac{d^2 \hat{S}_A}{d \hat{\Theta}^2},$$

and it easy to show that α^Θ times the right-hand side of Eqn. (A.23.1) is equal to β^Θ times the right-hand side of Eqn. (A.23.2).

Potential density ρ^θ or ρ^Θ

Potential density ρ^θ is the density that a fluid parcel would have if its pressure were changed to a fixed reference pressure p_r in an isentropic and isohaline manner. Potential density referred to reference pressure p_r can be written as

$$\rho^\theta(S_A, t, p, p_r) = \rho(S_A, \theta[S_A, t, p, p_r], p_r) = g_p^{-1}(S_A, \theta[S_A, t, p, p_r], p_r). \quad (3.4.2)$$

Using the functional forms of either $\rho = \tilde{\rho}(S_A, \theta, p)$ or $\rho = \hat{\rho}(S_A, \Theta, p)$, potential density with respect to reference pressure p_r (e. g. 1000 dbar) can be evaluated more easily as

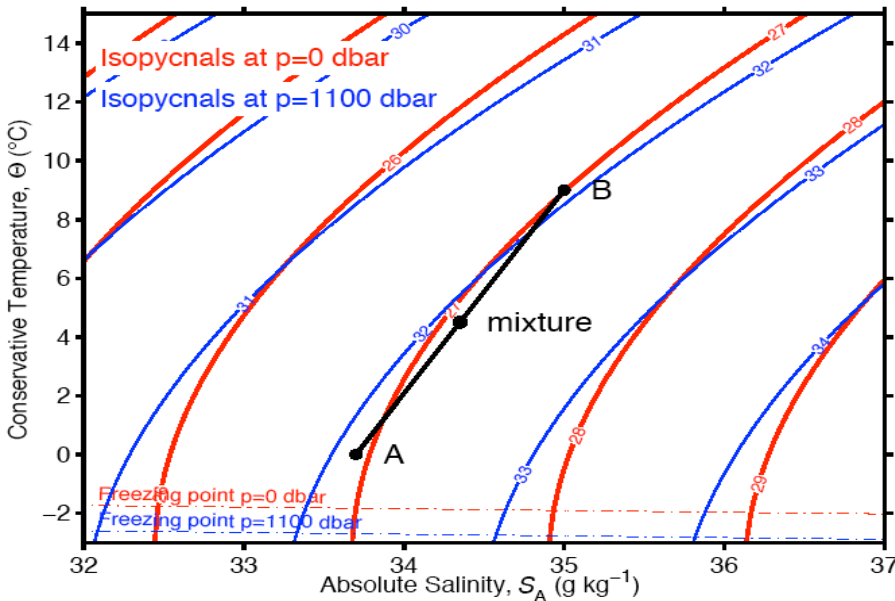
$$\rho^\theta(S_A, t, p, p_r) = \tilde{\rho}(S_A, \eta, p_r) = \tilde{\rho}(S_A, \theta, p_r) = \hat{\rho}(S_A, \Theta, p_r), \quad (3.4.3)$$

where we note that the potential temperature θ in the penultimate expression is the potential temperature with respect to 0 dbar. Once the reference pressure is fixed, potential density is a function only of Absolute Salinity and Conservative Temperature (or equivalently, of Absolute Salinity and potential temperature). Note that it is equally correct to label potential density as ρ^θ or ρ^Θ (or indeed as ρ^η) because η , θ and Θ are constant during the isentropic and isohaline pressure change from p to p_r ; that is, these variables possess the “potential” property.

Since we know that $v = g_p = \tilde{h}_p = \hat{h}_p$, potential density may also be expressed in terms of the pressure derivative of the expressions $h = \tilde{h}(S_A, \theta, p)$ and $h = \hat{h}(S_A, \Theta, p)$ as

$$\rho^\theta(S_A, t, p, p_r) = [\tilde{h}_p(S_A, \theta, p = p_r)]^{-1} = [\hat{h}_p(S_A, \Theta, p = p_r)]^{-1}. \quad (3.4.4)$$

The figure below shows contours of constant potential density on the $S_A - \Theta$ diagram. The red contours have $p_r = 0$ dbar while the blue contours are with respect to the reference pressure $p_r = 1100$ dbar. Notice particularly that on this $S_A - \Theta$ diagram the contours rotate with increasing reference pressure.



An important consequence of this rotation can be deduced by considering the two seawater parcels A and B. If parcels A and B were at the sea surface (that is, at 0 dbar) then parcel B would be denser than parcel A. However, if both seawater parcels were at 1100 dbar then the reverse is the case; now parcel A is denser than parcel B. Therein lies a whole level of complication in physical oceanography, all caused by the thermobaric non-linearity of the equation of state of seawater.

The symbol σ_1 is used for $\hat{\rho}^\Theta(S_A, \Theta, p_r=1000 \text{ dbar}) - 1000 \text{ kg m}^{-3}$ and similarly for σ_0, σ_2 and σ_4 , and these are called “potential density anomaly”.

Here is another figure illustrating the rotation of the potential density contours with pressure.

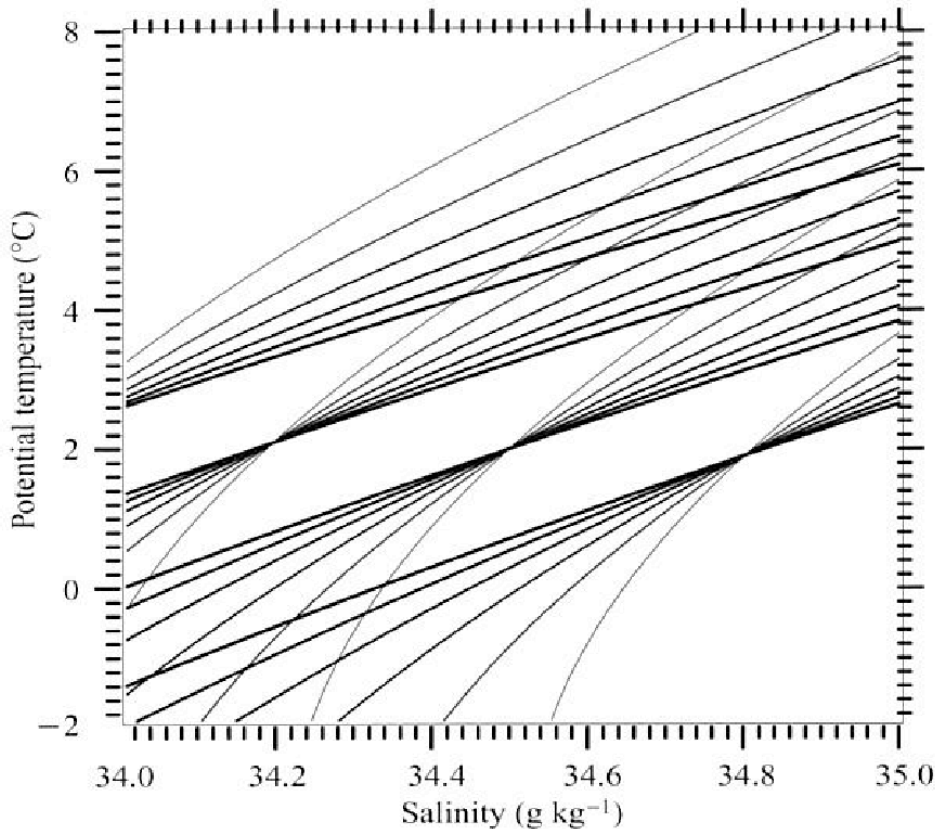


Figure 2.1. Equation of state for sea water. Contours of the density difference $\rho(p, \theta, S) - \rho(p, 2^\circ\text{C}, 34.5 \text{ psu})$ are shown in the (θ, S) -plane for different values of pressure corresponding to depths of 0 m (*thin lines*) to 5 km (*thick lines*) in 1 km intervals. The contour interval is 0.25 kg m^{-3} . The equation of state is nonlinear. The contours (isopycnals) are *curved* and their slope *turns* with pressure. Courtesy of Ernst Maier-Reimer.

Calculating the thermodynamic properties of seawater using the GSW Oceanographic Toolbox

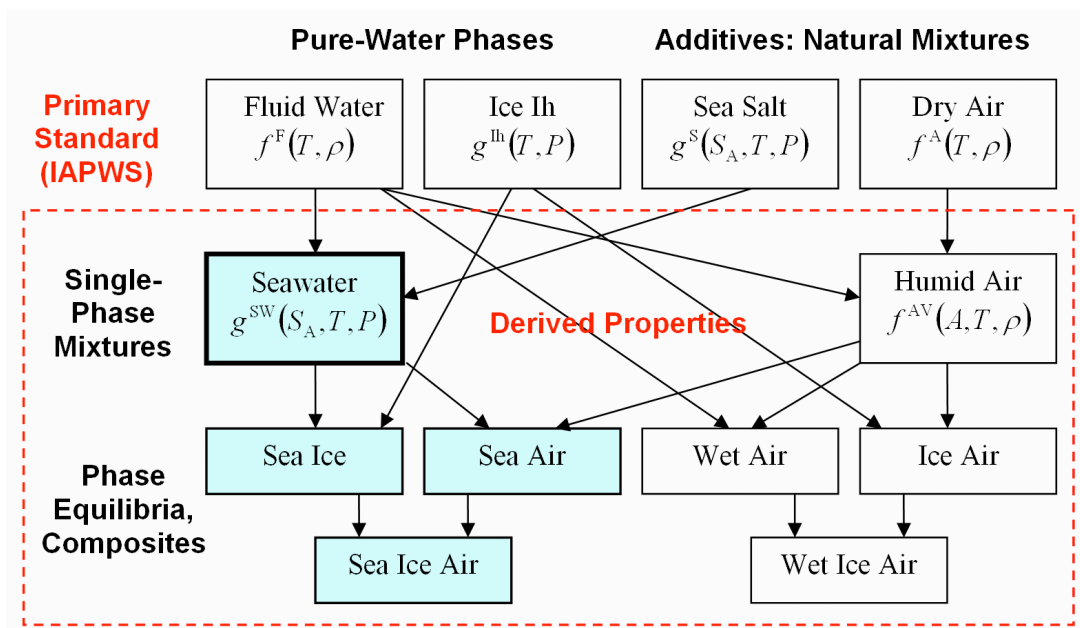
The computer software needed to evaluate the thermodynamic properties of seawater is available from the web site www.TEOS-10.org. The GSW Oceanographic Toolbox (GSW stands for Gibbs SeaWater) can be downloaded in a variety of computer languages. The above plot came from this GSW package. The list http://www.teos-10.org/pubs/GSW_Toolbox_list.pdf lists all the available algorithms.

For the past thirty years we have taken the “raw” data of Practical Salinity S_p (PSS-78), *in situ* temperature t (now ITS-90) and pressure p and we have used an algorithm to calculate potential temperature θ in order to analyze and publish water-mass characteristics on the $S_p - \theta$ diagram. On this $S_p - \theta$ diagram we have been able to draw curved contours of potential density using EOS-80 which has been the international standard for seawater from 1980-2009.

Under TEOS-10 this practice has now changed:- density and potential density (and all types of geostrophic streamfunction including dynamic height anomaly) are now not functions of Practical Salinity S_p but rather are functions of Absolute Salinity S_A .

In summary, under EOS-80 we have used the observed variables (S_p, t, p) to first form potential temperature θ and then we have analyzed water masses on the $S_p - \theta$ diagram, and we have been able to draw curved contours of potential density on this same $S_p - \theta$ diagram. Under TEOS-10, the observed variables (S_p, t, p) , together with longitude and latitude, are used to first form Absolute Salinity S_A using `gsw_SA_from_SP`, and then Conservative Temperature Θ is calculated using `gsw_CT_from_t`. Oceanographic water masses are then analyzed on the $S_A - \Theta$ diagram (using `gsw_SA_CT_plot`), and potential density contours can be drawn on this $S_A - \Theta$ diagram using `gsw_rho(SA,CT,p_ref)`.

So the first steps with analyzing observed oceanographic data is to calculate and store Absolute Salinity S_A and Conservative Temperature Θ . Thereafter, all the analysis uses these variables and does not make any further use of the observed Practical Salinity S_p , nor of the *in situ* temperature t , nor of the potential temperature θ .



Formulas for properties of seawater and ice expressed in terms of the Gibbs functions $g(S_A, T, p)$ for seawater and $g(T, p)$ for ice.

Property	Symbol	Expression in $g(S, T, p)$ of seawater	Expression in $g(T, p)$ of ice
specific Gibbs energy	g	g	g
specific enthalpy	h	$g - T g_T$	$g - T g_T$
specific Helmholtz energy	f	$g - p g_p$	$g - p g_p$
specific internal energy	u	$g - T g_T - p g_p$	$g - T g_T - p g_p$
Specific entropy	s	$-g_T$	$-g_T$
pressure	p	p	p
density	ρ	$1 / g_p$	$1 / g_p$
specific isobaric heat capacity	c_p	$-T g_{TT}$	$-T g_{TT}$
thermal expansion	α	g_{Tp} / g_p	g_{Tp} / g_p
isothermal compressibility	κ_T	$-g_{pp} / g_p$	$-g_{pp} / g_p$
isentropic compressibility	κ_s	$(g_{ip}^2 - g_{ip} g_{pp}) / (g_p g_{ii})$	$(g_{ip}^2 - g_{ip} g_{pp}) / (g_p g_{ii})$
Sound speed	w	$g_p \sqrt{g_{ii} / (g_{ip}^2 - g_{ip} g_{pp})}$	—
chemical potential of water	μ_w	$g - S_A g_S$	g
pressure coefficient for ice	β	—	$-g_{Tp} / g_{pp}$

Freezing temperature and isobaric melting enthalpy

As an example of the use of more than one of the above thermodynamic potentials, consider the process of the melting (or freezing) of ice into seawater.

Thermodynamic theory tells us that freezing occurs at the temperature t_f at which the chemical potential of water in seawater μ^w equals the chemical potential of ice μ^{lh} . Thus, t_f is found by solving the implicit equation

$$\mu^w(S_A, t_f, p) = \mu^{lh}(t_f, p) \quad (3.33.1)$$

or equivalently, in terms of the two Gibbs functions,

$$g(S_A, t_f, p) - S_A g_{S_A}(S_A, t_f, p) = g^{lh}(t_f, p). \quad (3.33.2)$$

The Gibbs function for ice Ih, $g^{lh}(t, p)$, is defined as part of TEOS-10, so we have very accurate freezing temperatures which are functions of Absolute Salinity and pressure.

Knowledge of the Gibbs functions of seawater and of ice also lead to very accurate values for the latent heat of melting (isobaric melting enthalpy), namely

$$L_p^{Sl}(S_A, p) = h - S_A \left. \frac{\partial h}{\partial S_A} \right|_{T, p} - h^{lh} = h - S_A (\mu - T \mu_T) - h^{lh}, \quad (3.34.6)$$

which is actually the difference between the partial specific enthalpies of water in seawater and of ice.

The vertical gradient of potential density

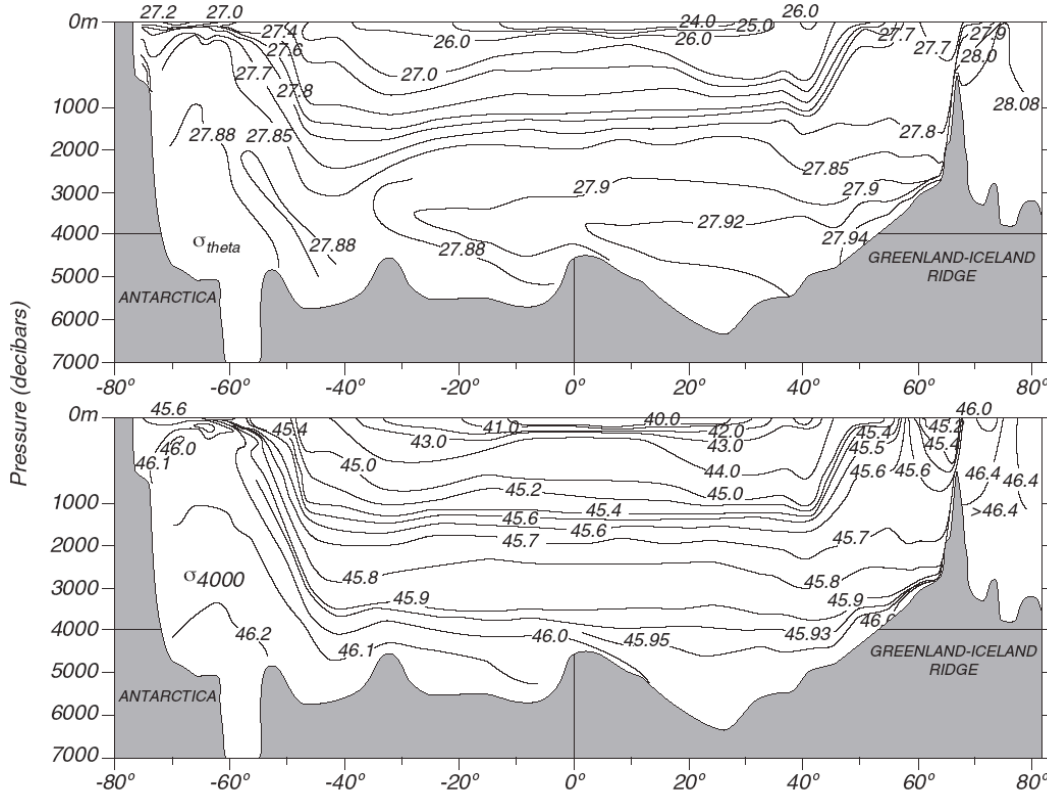


Figure 6.10 Vertical sections of density in the western Atlantic. Note that the depth scale changes at 1000 m depth. **Upper:** σ_θ , showing an apparent density inversion below 3,000 m. **Lower:** σ_4 showing continuous increase in density with depth. After Lynn and Reid (1968).

The potential density of a seawater sample (S_A, Θ, p) , referenced to reference pressure p_r is given by $\rho^\ominus(S_A, \Theta) = \hat{\rho}(S_A, \Theta, p_r)$. The vertical gradient of the natural logarithm of potential density is $\beta^\ominus(p_r)$ times the vertical gradient of Absolute Salinity minus $\alpha^\ominus(p_r)$ times the vertical gradient of Conservative Temperature,

$$\frac{1}{\rho^\ominus} \frac{\partial \rho^\ominus}{\partial z} = \beta^\ominus(p_r) S_{A_z} - \alpha^\ominus(p_r) \Theta_z. \quad (\text{A.26.2})$$

The ratio of this vertical gradient of potential density to the square of the buoyancy frequency is given by (Tutorial exercise)

$$\frac{-g \rho^{-1} \rho_z^\ominus}{N^2} = \frac{\beta^\ominus(p_r) [R_p/r - 1]}{\beta^\ominus(p) [R_p - 1]} = \frac{\beta^\ominus(p_r)}{\beta^\ominus(p)} \frac{1}{G^\ominus} \approx \frac{1}{G^\ominus}, \quad (3.20.5)$$

where r is the ratio of the slope on the $S_A - \Theta$ diagram of an isoline of potential density with reference pressure p_r to the slope of a potential density surface with reference pressure p , and is defined by

$$r = \frac{\alpha^\ominus(S_A, \Theta, p) / \beta^\ominus(S_A, \Theta, p)}{\alpha^\ominus(S_A, \Theta, p_r) / \beta^\ominus(S_A, \Theta, p_r)}, \quad (3.17.2)$$

and the “isopycnal temperature gradient ratio” G^\ominus is defined by

$$G^\ominus \equiv \frac{[R_p - 1]}{[R_p/r - 1]} \quad \text{where} \quad R_p = \frac{\alpha^\ominus \Theta_z}{\beta^\ominus(S_A)_z} \quad (3.17.4)$$

is the ratio of the vertical contribution from Conservative Temperature to that from Absolute Salinity to the static stability N^2 of the water column. The name “isopycnal temperature gradient ratio” is chosen for G^\ominus because it can be

shown that G^Θ is the ratio of the gradient of Conservative Temperature in a potential density surface to that in a neutral tangent plane (Tutorial exercise),

$$\nabla_\sigma \Theta = G^\Theta \nabla_n \Theta . \quad (3.17.3)$$

The saline contraction coefficient $\beta^\Theta(S_A, \Theta, p)$ does not vary very much from a constant value compared with variation of the thermal expansion coefficient $\alpha^\Theta(S_A, \Theta, p)$. That is, you make a 10% - 20% error by approximating r as

$$r \approx \frac{\alpha^\Theta(S_A, \Theta, p)}{\alpha^\Theta(S_A, \Theta, p_t)} . \quad (3.17.2_approx)$$

There is never any reason to actually make this approximation in numerical work, rather this approximation can aid in thinking about what causes what in the ocean. [You can check that this is a good approximation by inspection of the red and blue potential density contours on the above $S_A - \Theta$ diagram.]

Also, the slope difference between that of a neutral tangent plane and a potential density surface is given by (Tutorial exercise)

$$\begin{aligned} \nabla_n z - \nabla_\sigma z &= \frac{\nabla_n \Theta - \nabla_\sigma \Theta}{\Theta_z} = (1 - G^\Theta) \frac{\nabla_n \Theta}{\Theta_z} \\ &= \frac{R_\rho [1-r]}{[R_\rho - r]} \frac{\nabla_n \Theta}{\Theta_z} = \frac{R_\rho [1-r]}{r [R_\rho - 1]} \frac{\nabla_\sigma \Theta}{\Theta_z} . \end{aligned} \quad (3.18.1)$$

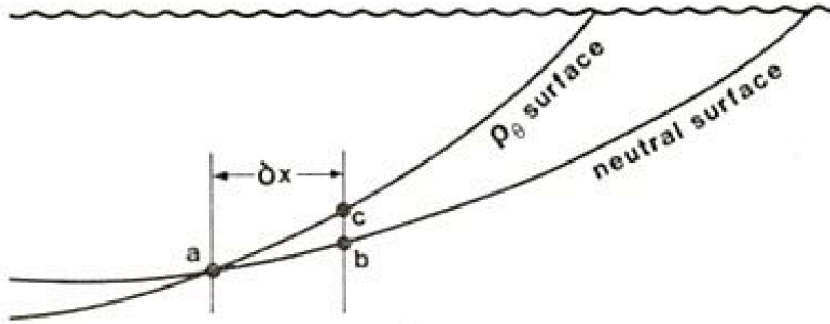


FIG. 1. Sketch of a cross section through the ocean showing a neutral surface and a potential density surface passing through point a. At a horizontal distance δx from point a, a vertical cast cuts the two surfaces at points b and c.

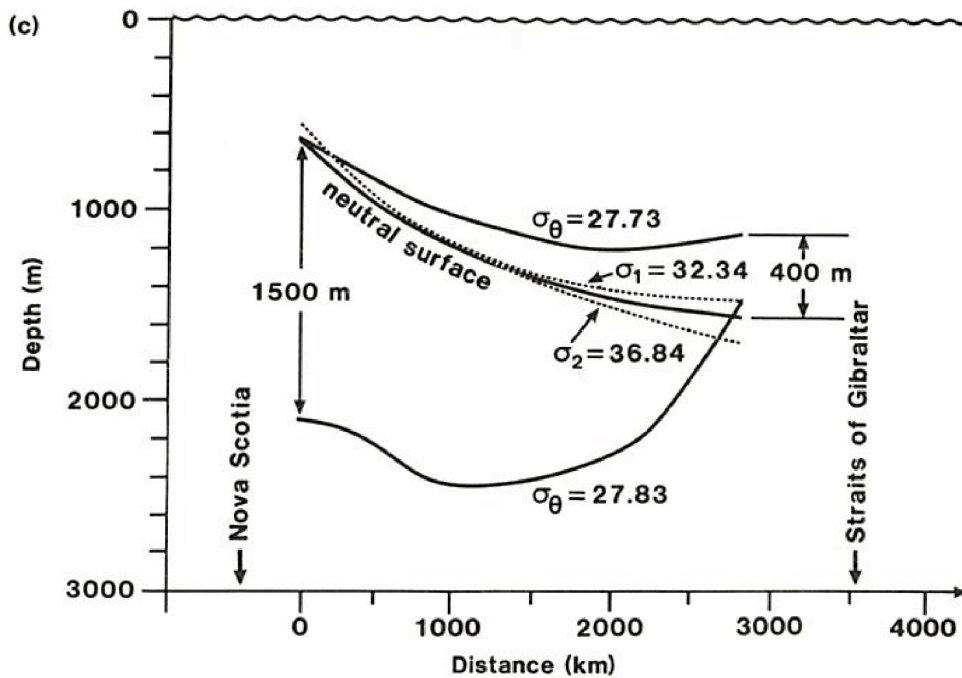
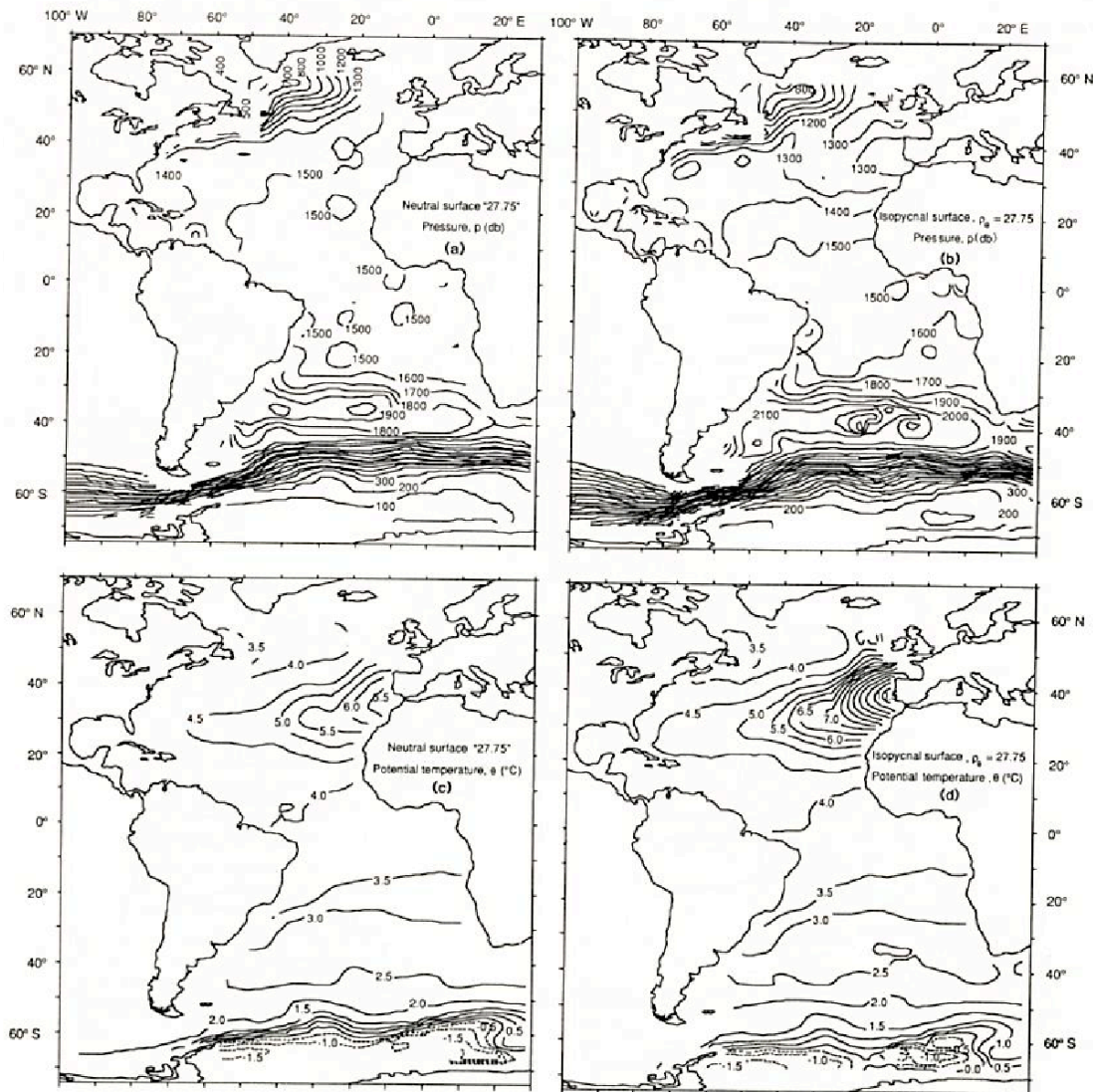


FIG. 7. Maps of pressure on two potential density surfaces: (a) $\sigma_\theta = 27.73$; (b) $\sigma_\theta = 27.83$. The potential density surfaces intersect the same neutral surface (NSa of Fig. 6) at different positions. This is illustrated in cross section in (c), which goes from near Nova Scotia on the left to near the Straits of Gibraltar on the right. Also shown (dashed lines) are a potential density surface referenced to a pressure of 1000 db ($\sigma_1 = 32.34$) and a potential density surface referenced to 2000 db ($\sigma_2 = 36.84$).



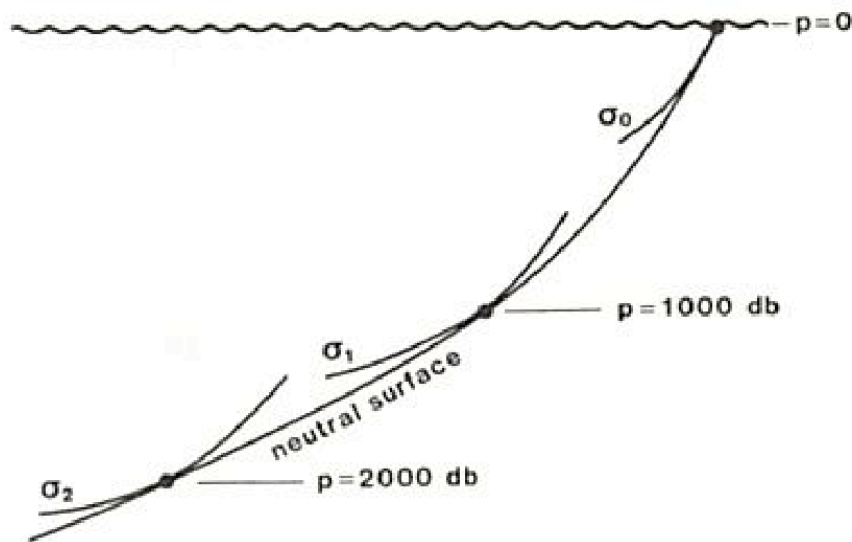


FIG. 2. Sketch of a neutral surface and three different potential density surfaces, referenced to 0 db, 1000 db and 2000 db. The neutral surface is tangential to potential density surfaces only at the reference pressure of those potential density surfaces. In this way, the neutral surface can be regarded as the envelope curve of many locally referenced potential density surfaces with continually changing reference pressures. The definition of a neutral surface adopted in this paper avoids the concept of potential density and in particular, avoids the changing reference pressure which is endemic to a neutral surface defined in terms of potential density concepts.

